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MEMORANDUM

SOURCE DISTRIBUTION METHOD FOR UNSTEADY ONE-DIMENSIONAL

FLOWS WITH SMALL MASS, MOMENTUM, AND HEAT

ADDITION AND SMALL AREA VARIATION

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TABLE OF CONTENTS

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SOURCE DISTRIBUTION METHOD FOR UNSTEADY ONE-DIMENSIONAL FLOWS WITH SMALL MASS, MOMENTUM, AND HEAT ADDITION AND SMALL AREA VARIATION

By Harold Mirels

SUMMARY

A source distribution method is presented for obtaining flow perturbations due to small unsteady area variations, mass, momentum, and heat additions in a basic uniform (or piecewise uniform) one-dimensional flow. First, the perturbations due to an elemental area variation, mass, momentum, and heat addition are found. The general solution is then represented by a spatial and temporal distribution of these elemental (source) solutions. Emphasis is placed on discussing the physical nature of the flow phenomena.

The method is illustrated by several examples. These include the determination of perturbations in basic flows consisting of (1) a shock propagating through a nonuniform tube, (2) a constant-velocity piston driving a shock, (3) ideal shock-tube flows, and (4) deflagrations initiated at a closed end. The method is particularly applicable for finding the perturbations due to relatively thin wall boundary layers.

INTRODUCTION

Numerous studies have been made of unsteady one-dimensional flows through ducts with area variation, body forces, mass, and heat addition. The most general method of handling such problems is by the method of characteristics which, for these problems, employs the "Riemann variables." A full account of this method is given in reference 1.

When the area variation, body forces, and so forth introduce only small disturbances into an otherwise uniform (or piecewise uniform) basic flow, the equations of motion can be readily linearized. As is usual with linear problems, many different methods and viewpoints can then be used to obtain solutions. One approach is to find first the perturbations associated with an elemental area change, body force, mass, and heat addition. The general linearized solution of an unsteady one-dimensional flow can then be represented by a spatial and temporal distribution of such elemental solutions. This approach may be termed a "source distribution method." The advantage of this viewpoint over more formal methods of solution is that it often gives a better physical insight into the flow processes. A discussion of the source distribution concept, as applied to one-dimensional unsteady flows, does not appear to exist explicitly in the current literature. Hence, such a discussion is presented herein. To emphasize the physical nature of these flows, the solutions are first deduced from a relatively simple flow problem. The solutions are then again deduced by formal manipulation of the equations of motion as they appear in the method of characteristics. The use of these solutions is then illustrated by several examples.

The source distribution method has been previously applied, by the author, to find nonuniformities in a shock tube due to the unsteady boundary layer along the shock tube wall (refs. 2 and 3). The present report may be considered as an extension and elaboration of the method presented in those reports.

ANALYSIS

A uniform basic flow is assumed to be slightly disturbed by small area changes, body forces, mass, and heat additions, which induce unsteady one-dimensional perturbations. The perturbations due to an elemental area change, body force, and so forth are first deduced from a simple flow problem. The general solution is then expressed as the superposition of these elemental flows. Finally, the general solution is again obtained by a formal linearization of the equations of motion as they appear in the method of characteristics.

Equations of Motion

Consider a uniform basic flow through a tube of constant area A. The uniform fluid properties are denoted by the symbols p, ρ, T, u, ... (Symbols are defined in appendix A.) Let perturbations from those uniform values be denoted by the prefix Δ so that the net pressure at a point is $p + \Delta p$, the net density is $\rho + \Delta\rho$, ... The perturbations are generally functions of (x,t) so that $\Delta p \equiv \Delta p(x,t)$, $\Delta\rho \equiv \Delta\rho(x,t)$, and so forth. By assuming that the perturbations are due to small area variation, body forces, mass, and heat additions, the equations of motion are:

Continuity:

$$\frac{\partial\Delta\rho}{\partial t} + u\,\frac{\partial\Delta\rho}{\partial x} + \rho\,\frac{\partial\Delta u}{\partial x} = \mu \tag{1a}$$

Momentum:

$$\frac{\partial \Delta u}{\partial t} + u \frac{\partial \Delta u}{\partial x} + \frac{1}{\rho} \frac{\partial \Delta p}{\partial x} = \frac{f}{\rho} \tag{1b}$$

Energy:

$$\frac{\partial \Delta s}{\partial t} + u \frac{\partial \Delta s}{\partial x} = \frac{q}{\rho T} \tag{1c}$$

State:

$$\frac{\Delta p}{p} - \frac{\Delta \rho}{\rho} - \frac{\Delta T}{T} = 0 \tag{1d}$$

where

$\mu(x,t) \equiv m - \frac{\rho}{A}\left(\frac{\partial \Delta A}{\partial t} + u \frac{\partial \Delta A}{\partial x}\right)$

$m(x,t) \equiv$ mass addition, per unit volume, per unit time

$\Delta A(x,t) \equiv$ perturbation of cross-sectional area

$f(x,t) \equiv$ body force, per unit volume, acting in $+x$ direction

$q(x,t) \equiv$ heat addition, per unit volume, per unit time

$\Delta s(x,t) \equiv c_v\left(\frac{\Delta p}{p} - \gamma \frac{\Delta \rho}{\rho}\right) \equiv$ entropy perturbation

The quantities μ, f, and q may be referred to as "volumetric sources." They are, respectively, sources of mass[1], momentum, and heat. The term "volumetric" stems from the fact that they are defined on a "per unit volume" basis.

For a given area variation $\Delta A(x,t)$, the average velocity (normal to the wall) of a fluid particle at the wall at any point (x,t) is given by

$$v = -\frac{1}{l}\left(\frac{\partial \Delta A}{\partial t} + u \frac{\partial \Delta A}{\partial x}\right)$$

[1]The quantity μ may be considered as the equivalent volumetric mass source in a tube of constant area, which induces the same perturbations as a prescribed mass addition m and area variation ΔA.

where l is the perimeter corresponding to the cross-sectional area A. The normal velocity is positive when directed inward. Then, for the case of area variation (but no external mass addition),

$$\mu = \frac{\rho l}{A} v \tag{2}$$

This expression is convenient for finding perturbations due to thin unsteady boundary layers along a constant-area tube. Here, v is the vertical velocity at the outer edge of the boundary layer, as computed from boundary-layer theory. This approach is used in references 2 and 3 to find nonuniformities in shock tubes due to the wall boundary layer.

Elemental Source Solutions

The quantities μ, f, and q were referred to as volumetric sources since they were defined on a per unit volume basis. Equivalent quantities, defined on a per unit cross-sectional area basis, may be referred to as "planar sources." For example, planar mass, momentum, and heat sources at $x = 0$ can be defined by the relations

$$\bar{\mu} \equiv \lim_{\substack{\epsilon \to 0 \\ \mu \to \infty}} \int_{-\epsilon}^{\epsilon} \mu \, dx \qquad \bar{f} \equiv \lim_{\substack{\epsilon \to 0 \\ f \to \infty}} \int_{-\epsilon}^{\epsilon} f \, dx \qquad \bar{q} \equiv \lim_{\substack{\epsilon \to 0 \\ q \to \infty}} \int_{-\epsilon}^{\epsilon} q \, dx \tag{3}$$

where

$\bar{\mu} \equiv$ equivalent mass addition at $x = 0$, per unit A, per unit time

$\bar{f} \equiv$ body force at $x = 0$, per unit A, in $+x$ direction

$\bar{q} \equiv$ heat addition at $x = 0$, per unit A, per unit time

In general, $\bar{\mu}$, $\bar{f}$, and $\bar{q}$ can be functions of time. In the present section, a simple flow containing planar sources will be considered. The solution of this flow will then be used to deduce the flow field induced by an elemental source.

Assume that planar mass, momentum, and heat sources are placed at $x = 0$ at time $t = 0$ (in an otherwise uniform flow) and that they remain at constant strength thereafter. The resulting perturbed flow field is indicated in figure 1 (for the case where the basic uniform flow is subsonic). In figure 1(b), line a is a downstream propagating acoustic wave moving with velocity $u + a$, line b is a contact surface between two regions of different entropy, moving with velocity u (neglecting perturbation velocities), and line c is an upstream propagating acoustic wave moving with velocity $u - a$. The lines a, b, c, and $x = 0$ subdivide the flow into four regions, 1, 2, 3, and 4. Region 1 is the original undisturbed uniform flow. The perturbations in regions 2, 3, and 4 are independent of x and t.

The magnitude of the perturbations in the various regions can be found by considering the jump in the perturbation quantities across lines a, b, c, and $x = 0$. The case $M < 1$ is treated here, while the case $M > 1$ is treated in appendix B.

Line a. - Line a is a downstream propagating acoustic wave. The acoustic relations then give

$$\Delta p_2 = \rho a \Delta u_2 \tag{4a}$$

$$\Delta \rho_2 = \frac{\rho}{a} \Delta u_2 \tag{4b}$$

Line b. - The fluid velocity and pressure are continuous across a contact surface so that

$$\Delta u_3 = \Delta u_2 \tag{5a}$$

$$\Delta p_3 = \Delta p_2 \tag{5b}$$

Line c. - Line c is an upstream propagating acoustic wave. The acoustic relations give

$$\Delta p_4 = -\rho a \Delta u_4 \tag{6a}$$

$$\Delta \rho_4 = -\frac{\rho}{a} \Delta u_4 \tag{6b}$$

Line $x = 0$. - Integrating equations (1a) to (1c) across $x = 0$ and noting equations (3) give, respectively,

$$u(\Delta \rho_3 - \Delta \rho_4) + \rho(\Delta u_3 - \Delta u_4) = \overline{\mu} \tag{7a}$$

$$u(\Delta u_3 - \Delta u_4) + \frac{1}{\rho}(\Delta p_3 - \Delta p_4) = \frac{\overline{f}}{\rho} \tag{7b}$$

$$u\left(\frac{\Delta p_3}{p} - \gamma \frac{\Delta \rho_3}{\rho}\right) = \frac{\overline{q}}{c_V \rho T} \tag{7c}$$

Equation (7c) follows from $\Delta s_4 = 0$, $\Delta s_3 = c_V\left[(\Delta p_3/p_3) - \gamma(\Delta \rho_3/\rho_3)\right]$.

Equations (4) to (7) are nine equations in nine unknowns. The solution can be expressed as

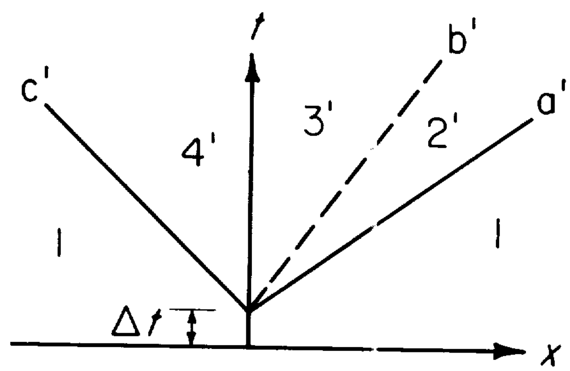
$$\frac{\Delta p_2}{p} = \frac{\gamma}{2(1 + M)\rho a}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} + \frac{\bar{f}}{a}\right) \tag{8a}$$

$$\frac{\Delta p_4}{p} = \frac{\gamma}{2(1 - M)\rho a}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} - \frac{\bar{f}}{a}\right) \tag{8b}$$

$$\frac{\Delta s_3}{c_v} = \frac{\bar{q}}{c_v \rho u T} \tag{8c}$$

All other perturbations are then found directly from equations (4) to (7).

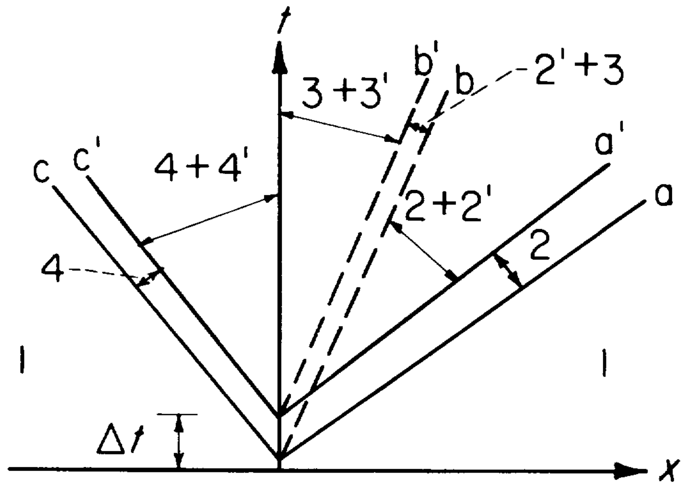
Now, assume that equal and opposite (with respect to the previous example) planar mass, momentum, and heat sources are placed at $x = 0$ at time $t = \Delta t$ (in the same otherwise uniform flow) and that they remain at constant strength thereafter. The corresponding t-x diagram is shown in sketch (a). The lines representing the acoustic waves and



(a)

the contact surface are displaced upward by an amount Δt from the corresponding lines in figure 1(b). The perturbations in regions 2', 3', and 4' are equal and opposite to those of the corresponding regions in figure 1(b).

If the flow fields in figure 1(b) and sketch (a) are linearly superposed, the resulting flow (sketch (b)) has zero perturbations everywhere except between lines a, a'; b, b'; and c, c' (since the perturbations in regions 2, 3, and 4 are equal and opposite to those in regions 2', 3', and 4'). By using a superscript $(\ )^+$ to denote perturbations due to a downstream propagating acoustic wave, the pressure perturbations between lines a and a' can be written (from eq. (8a))



(b)

$$\frac{\Delta p^+}{p} = \frac{\gamma}{2(1 + M)\rho a}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} + \frac{\bar{f}}{a}\right) \tag{9a}$$

Also, for this region (from eqs. (4)), $(\Delta u^+/a) = (\Delta\rho^+/\rho) = (\Delta p^+/\gamma p)$. By denoting the perturbations due to an upstream propagating acoustic wave by the superscript $(\ )^-$, the pressure perturbation between lines c and c' can be written (from eq. (8b))

$$\frac{\Delta p^-}{p} = \frac{\gamma}{2(1 - M)\rho a}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} - \frac{\bar{f}}{a}\right) \tag{9b}$$

Also, for this region (from eqs. (6)), $(-\Delta u^-/a) = (\Delta p^-/\rho) = (\Delta p^-/\gamma p)$. Between lines b and b', the velocity and pressure perturbations are zero, while the entropy perturbation is given by (from eq. (8c))

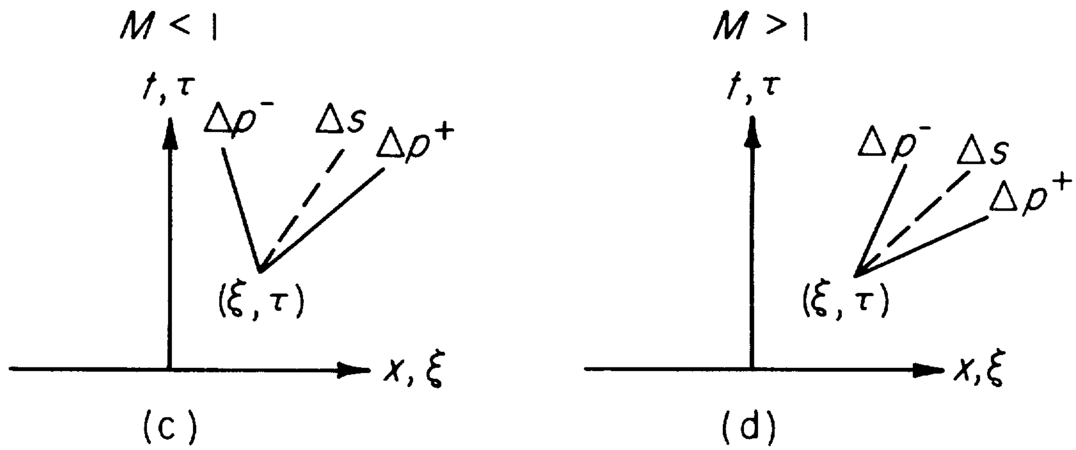
$$\frac{\Delta s}{c_v} = \frac{\bar{q}}{c_v \rho u T} \tag{9c}$$

The corresponding density perturbation is $(\Delta\rho/\rho) = -(\Delta s/c_p)$.

A similar development is presented in appendix B for the $M > 1$ case. The results for Δp^+, Δp^-, and Δs are the same as for the $M < 1$ case (e.g., eqs. (9)) except that the expression for Δp^- contains the term $(M - 1)$ for the $M > 1$ case. Use of $|1 - M|$ in the demonimator of equation (9b) makes equations (8) applicable for both $M < 1$ and $M > 1$.

Now, consider sketch (b) in the limit as $\Delta t \to 0$. It is then clear that a planar source at $x = 0$, $t = 0$ instantaneously generates a downstream propagating acoustic wave, an upstream propagating acoustic wave, and an entropy wave which propagates along the lines $x = (u + a)t$, $x = (u - a)t$, and $x = ut$, respectively. The strength of these waves is given by equations (9). <u>The perturbations are zero everywhere except on these lines</u>.

The results of the previous paragraph may be readily generalized to the case where the planar source is placed at the arbitrary point $x = \xi$, $t = \tau$ (see sketches (c) and (d)):



The resulting perturbations and the lines along which these perturbations propagate are then

$$\left.\begin{aligned}\frac{\Delta p^+}{p} &= \frac{\gamma}{2(1 + M)\rho a}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} + \frac{\bar{f}}{a}\right)\\ &= \frac{\gamma \Delta u^+}{a} = \frac{\gamma \Delta \rho^+}{\rho}\end{aligned}\right\} \tag{10a}$$

along $x = \xi + (u + a)(t - \tau)$,

$$\left.\begin{aligned}\frac{\Delta p^-}{p} &= \frac{\gamma}{2|1 - M|\rho a}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} - \frac{\bar{f}}{a}\right)\\ &= \frac{\gamma \Delta u^-}{a} = \frac{\gamma \Delta \rho^-}{\rho}\end{aligned}\right\} \tag{10b}$$

along $x = \xi + (u - a)(t - \tau)$, and

$$\left.\begin{aligned}\frac{\Delta s}{c_v} &= \frac{\bar{q}}{c_v \rho u T} \\ &= \frac{-\gamma \Delta \rho}{\rho} \qquad (\Delta p = \Delta u = 0)\end{aligned}\right\} \tag{10c}$$

along $x = \xi + u(t - \tau)$.

The perturbations are zero everywhere except on these lines. The lines along which the disturbances propagate are termed "characteristic" lines. The line along which the entropy wave propagates may also be referred to as a "particle path" line since the entropy wave is convected by the stream.

Equations (10) are the elemental source solutions which we had set out to obtain. From these equations it is seen that $\bar{\mu}$ and $\bar{q}$ generate upstream and downstream propagating acoustic waves which have the same sign, whereas $\bar{f}$ generates upstream and downstream propagating acoustic waves which have opposite signs. The ratio of the downstream and upstream acoustic waves is $\Delta p^+/\Delta p^- = |1 - M|/(1 + M)$ for mass or heat addition and is $\Delta p^+/\Delta p^- = -|1 - M|/(1 + M)$ for a body force. Thus, the upstream propagating acoustic wave is stronger than the downstream wave (considering $\bar{\mu}$, $\bar{q}$, and $\bar{f}$ separately) except for $M = 0$ and $M \to \infty$. As M approaches 1, the value of Δp^- becomes very large (violating the assumption of small perturbations) and equations (10) become invalid.

General Solution

The perturbations due to an arbitrary spatial and temporal distribution of volumetric mass, momentum, and heat sources can be found by the linear superposition of the elemental solutions of equations (10). Again, use the coincident coordinate systems (ξ,τ) and (x,t) where (ξ,τ) defines the source location and (x,t) is the point at which the perturbations are to be found. Note that the volumetric mass source between $\xi - (d\xi/2)$ and $\xi + (d\xi/2)$ is equivalent to a planar mass source of strength $\bar{\mu}(\xi,\tau) = \mu(\xi,\tau)d\xi$, and so forth. Then, by linear superposition, the net perturbation at (x,t) due to an arbitrary volumetric source distribution is given by

$$\frac{\Delta p(x,t)}{p} = \frac{\Delta p^+(x,t)}{p} + \frac{\Delta p^-(x,t)}{p} \tag{11a}$$

$$\gamma \frac{\Delta u(x,t)}{a} = \frac{\Delta p^+(x,t)}{p} - \frac{\Delta p^-(x,t)}{p} \tag{11b}$$

$$\frac{\Delta\rho(x,t)}{\rho} = \frac{\Delta p(x,t)}{\gamma p} - \frac{\Delta s(x,t)}{c_p} \tag{11c}$$

$$\frac{\Delta T(x,t)}{T} = \frac{\Delta p(x,t)}{p} - \frac{\Delta\rho(x,t)}{\rho} \tag{11d}$$

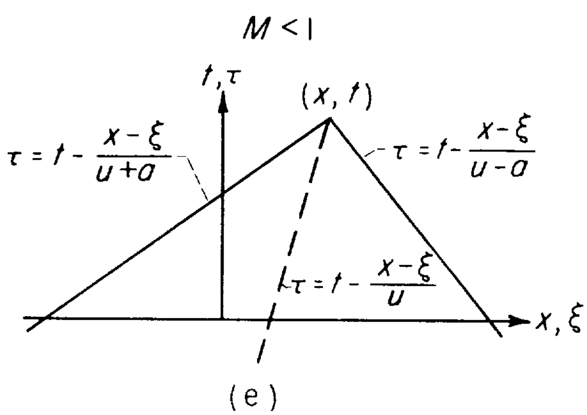
where

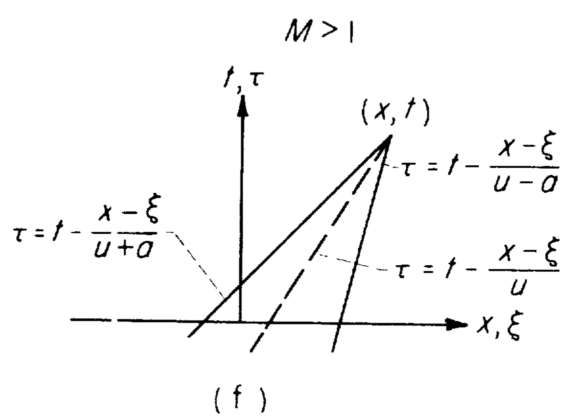
$$\frac{\Delta p^+(x,t)}{p} = \frac{\gamma}{2(1+M)\rho a}\int_{-\infty}^{x}\left\{\left[\mu(\xi,\tau) + \frac{q(\xi,\tau)}{c_pT} + \frac{f(\xi,\tau)}{a}\right]_{\tau=t-\frac{x-\xi}{u+a}}\right\}d\xi \tag{11e}$$

$$\frac{\Delta p^-(x,t)}{p} = \frac{\gamma}{2(1-M)\rho a}\int_{x}^{\pm\infty}\left\{\left[\mu(\xi,\tau) + \frac{q(\xi,\tau)}{c_pT} - \frac{f(\xi,\tau)}{a}\right]_{\tau=t-\frac{x-\xi}{u-a}}\right\}d\xi \tag{11f}$$

$$\frac{\Delta s(x,t)}{c_v} = \frac{1}{c_v\rho uT}\int_{-\infty}^{x}\left\{\left[q(\xi,\tau)\right]_{\tau=t-\frac{x-\xi}{u}}\right\}d\xi \tag{11g}$$

The integrations in equations (11e) to (11g) add the contributions of all the sources which contribute to Δp^+, Δp^-, and Δs at (x,t). These sources lie along the characteristic lines $\tau = t - \frac{x-\xi}{u \pm a}$ and $\tau = t - \frac{x-\xi}{u}$ as indicated in sketches (e) and (f):





The upper limit in equation (11f) is $(+\infty)$ or $(-\infty)$ depending on whether $M < 1$ or $M > 1$. (By using this convention, $d\xi/(1 - M)$ is always positive, and the absolute value sign is not needed for $1 - M$.)

Substitution of equations (11) into equations (1) verifies that they are indeed the general solution of equations (1). Equations (11) could have been deduced, formally, from equations (1). (This is, in fact, done in the next section.) The present development was undertaken so as to bring out more clearly the physical nature of the solution. Equations (11e) to (11g) assume that all perturbations are due only to the specified μ, q, and f distribution, so that there are no extraneous waves propagating in the tube. Otherwise, arbitrary functions of $x - (u + a)t$, $x - (u - a)t$, and $x - ut$ (i.e., homogeneous solutions of eqs. (1)) would have to be added to the right sides of equations (11e) to (11g), respectively. The quantities μ, q, and f must approach zero sufficiently fast, as ξ approaches $\pm\infty$, to make the integrals converge. As a result, the perturbations at $x = \pm\infty$ must be zero for all problems where equations (11) apply.

If the integrations in equations (11e) to (11g) are made with respect to τ instead of ξ, the integrals take on the form (since $d\tau/d\xi = 1/(u \pm a)$ for $\tau = t - (x - \xi)/(u \pm a)$ and $d\tau/d\xi = 1/u$ for $\tau = t - (x - \xi)/u$)

$$\frac{\Delta p^+(x,t)}{p} = \frac{\gamma}{2\rho}\int_{-\infty}^{t}\left\{\left[\mu(\xi,\tau) + \frac{q(\xi,\tau)}{c_pT} + \frac{f(\xi,\tau)}{a}\right]_{\xi=x-(u+a)(t-\tau)}\right\}d\tau \tag{12a}$$

$$\frac{\Delta p^-(x,t)}{p} = \frac{\gamma}{2\rho}\int_{-\infty}^{t}\left\{\left[\mu(\xi,\tau) + \frac{q(\xi,\tau)}{c_pT} - \frac{f(\xi,\tau)}{a}\right]_{\xi=x-(u-a)(t-\tau)}\right\}d\tau \tag{12b}$$

$$\frac{\Delta s}{c_v} = \frac{1}{c_v\rho T}\int_{-\infty}^{t}\left\{\left[q(\xi,\tau)\right]_{\xi=x-u(t-\tau)}\right\}d\tau \tag{12c}$$

Equations (12) are somewhat simpler than equations (11c) to (11g). The integrations always proceed in the $+\tau$ direction. Note that the coefficient $1 - M$ does not appear in equation (12b). Thus, Δp^- does not become infinite as $M \to 1$ (provided μ, q, and f approach zero sufficiently fast, as $\tau \to -\infty$), and equations (12) are generally applicable at $M = 1$ as well as $M \neq 1$.[2] (Eqs. (11) are also applicable at $M = 1$ provided $d\xi/(1 - M)$ is treated as an indeterminate form.)

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[2]This contrasts with the earlier result that $\Delta p^- \to \infty$ as $M \to 1$ for a planar source (eq. (10b)). This difference between the perturbations due to a planar source and a volumetric source is typical of source distributions in fluid flow problems. The intensity of the singularity induced by a source in a fluid flow field decreases when a point source is replaced by a surface distribution of sources and when a surface distribution is replaced by a volume distribution.

Alternate Derivation of General Solution

The general solution obtained in the previous section is rederived herein by a formal manipulation of the equations of motion as they appear in the method of characteristics (ref. 1). The latter employs the Riemann variables P and Q as dependent variables. The Riemann variables are then related to the dependent variables Δp^+ and Δp^-.

To solve one-dimensional flows of a perfect gas (with area change, body forces, mass, and heat addition) by the method of characteristics, the equations of motion can be written in the form (from eq. (III.d.9) of ref. 1)

$$\frac{\delta_+ P}{\delta t} = \frac{a}{2\rho}\left[\mu + \frac{q}{c_p T} + \frac{f}{a} + \frac{\rho}{c_p(\gamma - 1)}\frac{\delta_+ s}{\delta t}\right] \tag{13a}$$

$$\frac{\delta_- Q}{\delta t} = \frac{a}{2\rho}\left[\mu + \frac{q}{c_p T} - \frac{f}{a} + \frac{\rho}{c_p(\gamma - 1)}\frac{\delta_- s}{\delta t}\right] \tag{13b}$$

$$\frac{Ds}{Dt} = \frac{q}{\rho T} \tag{13c}$$

where

$$P = \frac{a}{\gamma - 1} + \frac{u}{2} \tag{13d}$$

$$Q = \frac{a}{\gamma - 1} - \frac{u}{2} \tag{13e}$$

$$\frac{\delta_\pm(\ )}{\delta t} = \frac{\partial(\ )}{\partial t} + (u \pm a)\frac{\partial(\ )}{\partial x} \tag{13f}$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} \tag{13g}$$

The quantities P and Q are the Riemann variables. (These equal one-half the values as defined in ref. 1.) Equations (13) define the variation of P, Q, and s in the characteristic directions $dx/dt = u + a$, $u - a$, and u, respectively. A numerical integration of these equations (together with the equations of state) can then be obtained by proceeding along the characteristic directions, in small increments, as discussed in reference 1.

If μ, f, and q are small, the equations can be linearized. Again, by letting the prefix Δ represent the departure of a flow variable from the basic uniform flow, equations (13) become

$$\gamma \frac{\delta_+}{\delta t}\left[\frac{\Delta P}{a} - \frac{\Delta s}{2c_p(\gamma - 1)}\right] = \frac{\gamma}{2\rho}\left(\mu + \frac{q}{c_pT} + \frac{f}{a}\right) \tag{14a}$$

$$\gamma \frac{\delta_-}{\delta t}\left[\frac{\Delta Q}{a} - \frac{\Delta s}{2c_p(\gamma - 1)}\right] = \frac{\gamma}{2\rho}\left(\mu + \frac{q}{c_pT} - \frac{f}{a}\right) \tag{14b}$$

$$\frac{D\Delta s}{Dt} = \frac{q}{\rho T} \tag{14c}$$

Integrating equations (14a) and (14b) along the characteristic directions gives (for $\Delta Q = \Delta P = \Delta s = 0$ at $t = -\infty$)

$$\gamma\left[\frac{\Delta P}{a} - \frac{\Delta s}{2c_p(\gamma - 1)}\right] = \frac{\gamma}{2\rho}\int_{-\infty}^{t}\left\{\left[\mu(\xi,\tau) + \frac{q(\xi,\tau)}{c_pT} + \frac{f(\xi,\tau)}{a}\right]_{\xi=x-(u+a)(t-\tau)}\right\}d\tau \tag{15a}$$

$$\gamma\left[\frac{\Delta Q}{a} - \frac{\Delta s}{2c_p(\gamma - 1)}\right] = \frac{\gamma}{2\rho}\int_{-\infty}^{t}\left\{\left[\mu(\xi,\tau) + \frac{q(\xi,\tau)}{c_pT} - \frac{f(\xi,\tau)}{a}\right]_{\xi=x-(u-a)(t-\tau)}\right\}d\tau \tag{15b}$$

The integral of equation (14c) is the same as equations (11g) or (12c). Comparison of equations (15) with equations (12a) and (12b) shows

$$\gamma\left[\frac{\Delta P}{a} - \frac{\Delta s}{2c_p(\gamma - 1)}\right] = \frac{\Delta p^+}{p} \tag{16a}$$

$$\gamma\left[\frac{\Delta Q}{a} - \frac{\Delta s}{2c_p(\gamma - 1)}\right] = \frac{\Delta p^-}{p} \tag{16b}$$

Thus, equations (12) (and eqs. (11e) to (11g)) could have been obtained by a formal integration of equations (13a) to (13c). Also, the elemental source solutions can be deduced from equations (15). However, it is felt that the previous derivation of these equations brings out the physical nature of the flow.

Equations (16) relate the perturbation of the Riemann variables ΔP and ΔQ to the variables of the previous section (namely Δp^+, Δp^-, and Δs). This relation can be deduced directly by a perturbation of the Riemann variables. Thus, perturbing equations (13d) and (13e) gives

$$\frac{\gamma}{a}\left\{\begin{matrix}\Delta P\\ \Delta Q\end{matrix}\right\} = \gamma\left(\frac{1}{\gamma - 1}\frac{\Delta a}{a} \pm \frac{1}{2}\frac{\Delta u}{a}\right) = \frac{\gamma}{2}\left\{\left[\frac{1}{\gamma}\frac{\Delta p}{p} + \frac{\Delta s}{c_p(\gamma - 1)}\right] \pm \frac{\Delta u}{a}\right\}$$

$$= \frac{\gamma}{2}\left\{\left[\frac{1}{\gamma}\left(\frac{\Delta p^+}{p} + \frac{\Delta p^-}{p}\right) + \frac{\Delta s}{c_p(\gamma - 1)}\right] \pm \frac{1}{\gamma}\left(\frac{\Delta p^+}{p} - \frac{\Delta p^-}{p}\right)\right\}$$

$$= \left\{\begin{matrix}\dfrac{\Delta p^+}{p} + \dfrac{\gamma\Delta s}{2c_p(\gamma - 1)}\\[2ex] \dfrac{\Delta p^-}{p} + \dfrac{\gamma\Delta s}{2c_p(\gamma - 1)}\end{matrix}\right\} \tag{17}$$

$$= \left\{\begin{matrix}\gamma\dfrac{\Delta u^+}{a} + \dfrac{\gamma\Delta s}{2c_p(\gamma - 1)}\\[2ex] -\gamma\dfrac{\Delta u^-}{a} + \dfrac{\gamma\Delta s}{2c_p(\gamma - 1)}\end{matrix}\right\} \tag{18}$$

Equations (17) agree with equations (16), as expected. Equations (17) and (18) provide a physical interpretation for ΔP and ΔQ within the limitation of linearized theory. These equations show that for isentropic flow (i.e., $\Delta s = 0$) the local values of $\gamma\Delta P/a$ and $\gamma\Delta Q/a$ exactly equal $\Delta p^+/p$ and $\Delta p^-/p$, respectively. Or, alternately, the local values of ΔP and ΔQ equal Δu^+ and $-\Delta u^-$, respectively. For $\Delta s \neq 0$, the relation between ΔP, ΔQ and Δp^+, Δp^- (or Δu^+, Δu^-) also depends on the local value of Δs and, therefore, depends on the thermal history of the fluid element at the section under consideration.

A linearized method of characteristics, employing ΔP, ΔQ, and Δs as the dependent variables, was developed in reference 4 for finding nonuniformities in a shock tube (due to the wall boundary layer). The method of reference 4 can be compared with the present method as follows. Reference 4, in effect, integrates equations (13a) to (13c) in the appropriate characteristic directions. Since s appears on the right side of equations (13a) and (13b), the solution for ΔP and ΔQ is coupled with the energy equation (eq. (13c)) for nonisentropic flows, and all three equations must be solved simultaneously. In the present method, Δp^+, Δp^-, and Δs are the dependent variables. However, the solution for Δp^+ and Δp^- is not coupled with the energy equation and can be found without a knowledge of Δs. Thus, the solution for Δp^+ and Δp^- is somewhat simpler than the corresponding solution for ΔP and ΔQ. In addition, Δp^+ and Δp^- have a simple physical interpretation, which is not the case for ΔP and ΔQ. For isentropic flow, the two methods become identical (except for notation).

APPLICATIONS

The applications of equations (11) and (12) are illustrated by considering several one-dimensional unsteady flow problems. In particular, three classes of problems are considered. The first class pertains to flows wherein the basic (unperturbed) flow extends from $x = -\infty$ to $x = +\infty$. The second class pertains to flows wherein there are two basic uniform flows separated by a discontinuity (such as a shock wave, contact surface, flame front, etc.) that moves with constant speed. The third class consists of several piecewise uniform basic flows, each separated by discontinuities moving with constant speed. For the latter class, attention is focused on cases wherein the discontinuities are centered. That is, they all originate at some fixed point, say $x = 0$, $t = 0$. Flow in an ideal shock tube (assuming the expansion wave has negligible thickness) is an example of a piecewise uniform flow with centered discontinuities.

Basic Flow Infinite in Extent

The solution of problems wherein the basic flow is infinite in extent is given directly by equations (11) and (12). Since the physical nature of this solution has already been discussed in detail, only two simple examples are treated herein.

First, consider a flow for which $\Delta A = \Delta A(\xi)$, $m = q = f = 0$. That is, the perturbations are due to a steady-state area variation in the tube. Integration of equations (11e) and (11f) gives (with $\mu = \frac{-\rho u}{A}\frac{d\Delta A(\xi)}{d\xi}$, $\Delta A(-\infty) = \Delta A(+\infty) = 0$)

$$\left.\begin{aligned} \frac{\Delta p^+(x)}{p} &= -\frac{\gamma}{2}\frac{M}{1+M}\frac{\Delta A(x)}{A} \\ \frac{\Delta p^-(x)}{p} &= +\frac{\gamma}{2}\frac{M}{1-M}\frac{\Delta A(x)}{A} \end{aligned}\right\} \tag{19}$$

which define the acoustic waves at station x. The net pressure and velocity perturbations at this section are then

$$\left.\begin{aligned} \frac{\Delta p(x)}{p} &= \frac{\gamma M^2}{1-M^2}\frac{\Delta A(x)}{A} \\ \gamma\frac{\Delta u(x)}{a} &= \frac{-\gamma M}{1-M^2}\frac{\Delta A(x)}{A} \end{aligned}\right\} \tag{20}$$

These are the same results as obtained directly from steady-state isentropic flow equations (e.g., ref. 5). Hence, steady flow through a slightly nonuniform tube can be viewed as the standing wave resulting from the superposition of unsteady acoustic waves generated by the source distribution $\mu = \frac{-\rho u}{A} \frac{d\Delta A(\xi)}{d\xi}$. Note that the perturbations at station x depend only on the local area perturbation $\Delta A(x)$ (eqs. (20)). The origin of these perturbations may be seen more clearly by considering a tube wherein the net area perturbation $\Delta A(x)$ occurs at station ξ_A and an equal and opposite area perturbation occurs at ξ_B, as indicated in figure 2(a). The equivalent planar source strength $\overline{\mu}$ at ξ_A and ξ_B is $-\rho u\Delta A(x)/A$ and $\rho u\Delta A(x)/A$, respectively. For a subsonic basic flow, the pressure perturbations at (x,t) are due to the Δp^+ wave originating from the source at (ξ_A,τ_A) and the Δp^- wave from the source at (ξ_B,τ_B) (fig. 2(b)). For a supersonic basic flow, the pressure perturbation at (x,t) is due to the Δp^+ wave from the source at (ξ_A,τ_A') and the Δp^- wave from the source at (ξ_A,τ_A'') (fig. 2(c)).

The second example is as follows: Consider a gas to be stationary and uniform in a tube of constant area. At time $\tau = 0$, volumetric body forces of unit strength are distributed along the tube from $\xi = 0$ to $\xi = 1$, and they remain at constant strength thereafter. That is,

$$f(\xi,\tau) = 0 \qquad \text{for all } \xi \text{ and } \tau < 0 \tag{21a}$$

$$\left.\begin{aligned} f(\xi,\tau) &= 0 \qquad \text{for } \xi < 0,\ \xi > 1 \\ &= 1 \qquad \text{for } 0 \le \xi \le 1 \end{aligned}\right\} \tau \ge 0 \tag{21b}$$

The problem is to find the resulting perturbations. These momentum sources occupy the crosshatched region in figure 3(a). The perturbations at a typical point (x,t) arise from those portions of the two characteristic lines through (x,t) which pass through the crosshatched region. (These portions are darkened in figure 3(a) for the two typical points (x,t) indicated therein.) From equations (11) it is seen that $\Delta p^+(x,t)$ is positive and is proportional to the length of the downstream propagating characteristic intersecting the crosshatched area. Also, $\Delta p^-(x,t)$ is negative and is proportional to the length of the upstream propagating characteristic intersecting the crosshatched area. As a result, each of the numbered regions in figure 3(b) has a different expression for the local net pressure perturbation. The pressure distribution in the tube at times $t = t'$ and $t = t''$ is indicated in figures 3(c) and (d). The solution is antisymmetric about $x = 1/2$. This type of approach is applicable for finding perturbations induced by impulsive application of a magnetic field to a conducting field flowing through a uniform tube.

Basic Flow Containing a Single Discontinuity

Consider a basic flow consisting of two regions, 1 and 2, separated by a discontinuity moving with velocity w (fig. 4(a)). The discontinuity may be a shock wave, contact surface, flame front, and so forth. The flow in regions 1 and 2 is assumed to be perturbed by mass, momentum, and heat sources. The problem is to find the perturbation at a typical point (x,t).

The net perturbation at any point is found by summing the contribution of all the elemental sources influencing the point. The sources contributing to a typical point in region 2 lie along the characteristics noted in figures 4(b) and (c) for the basic flows indicated therein. These characteristics are found in the following way. First, the downstream and upstream propagating acoustic characteristics and the particle path characteristic are drawn through (x,t). These are lines a, b, and c, respectively. These characteristics intersect the discontinuity at points B and C. All the possible characteristic lines in regions 1 and 2 which terminate at points B and C are then drawn.[3] This gives the additional characteristic lines used in figures 4(b) and (c). The physical significance of these lines is as follows. The sources along lines a, b, and c contribute directly to the perturbation at (x,t) in the manner discussed in the previous sections. The sources along line d generate a downstream propagating acoustic wave which arrives at (ξ_B,τ_B). The latter is designated $\Delta p_{2,B}^{+}$. This interacts with the discontinuity and generates a reflected wave (designated $\Delta p_{2,B}^{-}$) which propagates along line b so as to arrive at (x,t). Similarly, the sources along lines e, f, and j each generate a disturbance at point B which interacts with the discontinuity to contribute to the upstream propagating acoustic wave $\Delta p_{2,B}^{-}$. The sources along g, h, i, and k generate a disturbance at point C which interacts with the discontinuity so as to contribute to the entropy perturbation at point C in region 2 $(\Delta s_{2,C})$, which then propagates along line C so as to arrive at (x,t).

The ratio of the generated wave (at a discontinuity) to the incident wave is termed a "transfer function" herein. Acoustic reflection and transmission coefficients are special cases. The numerical value of the transfer function depends on the nature of the incident wave, the generated wave, and the discontinuity. Transfer functions for various situations are given in appendix C. For example, the following transfer functions must be known at point B in figure 4(b) in order to compute the perturbation at (x,t): $(\Delta p_{2,B}^{-}/p_2)/(\Delta p_{2,B}^{+}/p_2)$, $(\Delta p_{2,B}^{-}/p_2)/(\Delta p_{1,B}^{-}/p_1)$, and $(\Delta p_{2,B}^{-}/p_2)/(\Delta s_{1,B}/c_{v,1})$. The net perturbation $\Delta p_{2,B}^{-}/p_2$ is the sum

[3] P. 220 of ref. 12 gives all the possibilities.

of the contributions from the three incident waves noted in the denominators of the latter three transfer functions.

Formulas for the characteristic lines and intersection points in figures 4(b) and (c) are given as follows. The coordinate system is chosen so that the discontinuity goes through the origin.

Line a: $\xi = x - (u_2 + a_2)(t - \tau) \equiv \xi_a(\tau)$ $\qquad \tau = t - \left[(x - \xi)/(u_2 + a_2)\right] \equiv \tau_a(\xi)$ (22a)

Line b: $\xi = x - (u_2 - a_2)(t - \tau) \equiv \xi_b(\tau)$ $\qquad \tau = t - \left[(x - \xi)/(u_2 - a_2)\right] \equiv \tau_b(\xi)$ (22b)

Line c: $\xi = x - u_2(t - \tau) \equiv \xi_c(\tau)$ $\qquad \tau = t - \left[(x - \xi)/u_2\right] \equiv \tau_c(\xi)$ (22c)

Line d: $\xi = \xi_B - (u_2 + a_2)(\tau_B - \tau) \equiv \xi_d(\tau)$ $\qquad \tau = \tau_B - \left[(\xi_B - \xi)/(u_2 + a_2)\right] \equiv \tau_d(\xi)$ (22d)

Line e: $\xi = \xi_B - (u_1 - a_1)(\tau_B - \tau) \equiv \xi_e(\tau)$ $\qquad \tau = \tau_B - \left[(\xi_B - \xi)/(u_1 - a_1)\right] \equiv \tau_e(\xi)$ (22e)

Line f: $\xi = \xi_B - u_1(\tau_B - \tau) \equiv \xi_f(\tau)$ $\qquad \tau = \tau_B - \left[(\xi_B - \xi)/u_1\right] \equiv \tau_f(\xi)$ (22f)

Line j: $\xi = \xi_B - (u_1 + a_1)(\tau_B - \tau) \equiv \xi_j(\tau)$ $\qquad \tau = \tau_B - \left[(\xi_B - \xi)/(u_1 + a_1)\right] \equiv \tau_j(\xi)$ (22g)

Point B: $\xi_B = \left[\dfrac{x - (u_2 - a_2)t}{w - (u_2 - a_2)}\right]w$ $\qquad \tau_B = \dfrac{x - (u_2 - a_2)t}{w - (u_2 - a_2)}$. (22h)

Point C: $\xi_C = \dfrac{w(x - u_2 t)}{w - u_2}$ $\qquad \tau_C = \dfrac{x - u_2 t}{w - u_2}$ (22i)

The equations for lines g, h, i, and k are found from the equations for lines d, e, f, and j by replacing the subscript B with the subscript C.

The perturbations at (x,t), in figure 4(b), can now be expressed formally. Use the following notation for the integrals appearing in equations (12a) and (12b):

$$I^{+}_{2,\xi_a} \equiv \frac{\gamma_2}{2\rho_2}\int\left\{\left[\mu_2(\xi,\tau) + \frac{q_2(\xi,\tau)}{(c_pT)_2} + \frac{f_2(\xi,\tau)}{a_2}\right]_{\xi=\xi_a(\tau)}\right\}d\tau \qquad (23a)$$

$$I^{-}_{1,\xi_e} \equiv \frac{\gamma_1}{2\rho_1}\int\left\{\left[\mu_1(\xi,\tau) + \frac{q_1(\xi,\tau)}{(c_pT)_1} - \frac{f_1(\xi,\tau)}{a_1}\right]_{\xi=\xi_e(\tau)}\right\}d\tau \qquad (23b)$$

and so forth. The pressure perturbations at (x,t) can then be written (from eqs. (12) and fig. 4(b))

$$\frac{\Delta p^{+}_{2}(x,t)}{p_2} = \left(I^{+}_{2,\xi_a}\right)^{t}_{-\infty} \qquad (24a)$$

$$\frac{\Delta p^{-}_{2}(x,t)}{p_2} = \left(I^{-}_{2,\xi_b}\right)^{t}_{\tau_B} + \frac{\Delta p^{-}_{2,B}/p_2}{\Delta p^{+}_{2,B}/p_2}\left(I^{+}_{2,\xi_d}\right)^{\tau_B}_{-\infty} + \frac{\Delta p^{-}_{2,B}/p_2}{\Delta p^{-}_{1,B}/p_1}\left(I^{-}_{1,\xi_e}\right)^{\tau_B}_{-\infty} + \frac{\Delta p^{-}_{2,B}/p_2}{\Delta s_{1,B}/c_{v,1}}\frac{1}{(c_v\rho T)_1}\int_{-\infty}^{\tau_B}\left[q_1(\xi,\tau)\right]_{\xi=\xi_f}d\tau \qquad (24b)$$

The entropy perturbation at (x,t) is

$$\frac{\Delta s_2(x,t)}{c_{v,2}} = \frac{\Delta s_{2,C}}{c_{v,2}} + \frac{1}{(c_v\rho T)_2}\int_{\tau_C}^{t}\left[q_2(\xi,\tau)\right]_{\xi=\xi_c}d\tau \qquad (24c)$$

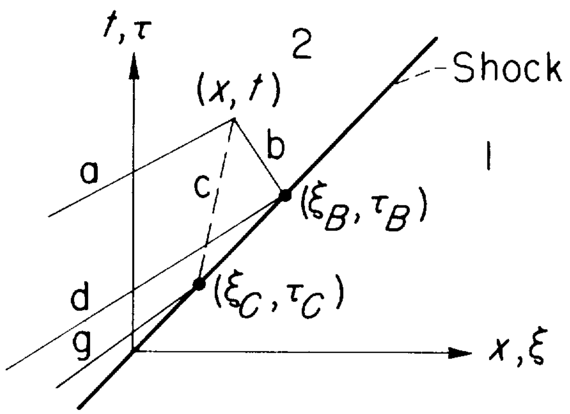
where

$$\frac{\Delta s_{2,C}}{c_{v,2}} = \frac{\Delta s_{2,C}/c_{v,2}}{\Delta p^{+}_{2,C}/p_2}\left(I^{+}_{2,\xi_g}\right)^{\tau_C}_{-\infty} + \frac{\Delta s_{2,C}/c_{v,2}}{\Delta p^{-}_{1,C}/p_1}\left(I^{-}_{1,\xi_h}\right)^{\tau_C}_{-\infty} + \frac{\Delta s_{2,C}/c_{v,2}}{\Delta s_{1,C}/c_{v,1}}\frac{1}{(c_v\rho T)_1}\int_{-\infty}^{\tau_C}\left[q_1(\xi,\tau)\right]_{\xi=\xi_i}d\tau$$

The net pressure, velocity, density, and temperature perturbations at (x,t) can then be found from equations (11a) to (11d).

The expressions for the perturbations at (x,t) in figure 4(c) require two additional terms because of characteristic lines j and k. The perturbations at a point in region 1 are found in a similar fashion.

As an example, consider the propagation of a shock wave in a duct having small area variations. This problem has been discussed, from other viewpoints, in references 6 to 8. The wave diagram for obtaining the perturbations behind the shock (region 2) is indicated in sketch (g). The fluid in region 1 is assumed to be uniform and at rest so that no disturbances originate in that region. Thus, with $\Delta A = \Delta A(\xi)$, $m = q = f = 0$, and $\Delta A(-\infty) = 0$, equations (24a) and (24b) give



(g)

$$\frac{\Delta p_2^+(x,t)}{p_2} = \left(I_{2,\xi_a}^+\right)_{-\infty}^{t} = \frac{-\gamma_2 M_2}{2(1 + M_2)} \frac{\Delta A(x)}{A} \tag{25a}$$

$$\frac{\Delta p_2^-(x,t)}{p_2} = \left(I_{2,\xi_b}^-\right)_{\tau_B}^{t} + \frac{\Delta p_{2,B}^-/p_2}{\Delta p_{2,B}^+/p_2}\left(I_{2,\xi_d}^+\right)_{-\infty}^{\tau_B}$$

$$= \frac{\gamma_2 M_2}{2(1 - M_2)}\left[\frac{\Delta A(x)}{A} - \frac{\Delta A(\xi_B)}{A}\right] + \frac{\Delta p_{2,B}^-/p_2}{\Delta p_{2,B}^+/p_2}\left[\frac{-\gamma_2 M_2}{2(1 + M_2)} \frac{\Delta A(\xi_B)}{A}\right] \tag{25b}$$

where $(\Delta p_{2,B}^-/p_2)/(\Delta p_{2,B}^+/p_2)$ is found from equation (C10a). The entropy perturbation at (x,t) is found by evaluating equation (25a) at (ξ_C, τ_C), which gives $\Delta p_{2,C}^+/p_2$, and multiplying by the transfer function $(\Delta s_{2,C}/c_{v,2})/(\Delta p_{2,C}^+/p_2)$, as given by equation (C10b). The present solution is in agreement with the previous treatments of this problem.

Basic Flow Containing Centered Discontinuities

Basic flows which contain a number of discontinuities, but are uniform between these discontinuities, can be treated by further extending the methods of the previous sections. In the present section, examples are considered wherein the discontinuities are centered at $x = 0$, $t = 0$.

The first example is that of a piston starting impulsively from rest and moving with uniform velocity u_p thereafter (fig. 5). The fluid in the tube is initially at rest (region 1). A shock is generated such that the fluid between the piston and the shock moves with a basic velocity equal to that of the piston (i.e., $u_2 = u_p$). The fluid in region 2 is assumed to be perturbed by mass, momentum, and heat sources, and the resulting perturbations are desired. No sources are assumed in region 1. The characteristics influencing a typical point in region 2 are indicated in figure 5(b). There are an infinite number of characteristic line segments due to successive wave reflections at the shock and piston. The expressions for the pressure perturbation at (x,t) are of the form:

$$\frac{\Delta p_2^+(x,t)}{p_2} = \left(I_{2,\xi_a}^+\right)_{\tau_A}^{t} + \left(\frac{\Delta p_{2,A}^+/p_2}{\Delta p_{2,A}^-/p_2}\right)\left\{\left(I_{2,\xi_e}^-\right)_{\tau_e}^{\tau_A} + \frac{\Delta p_{2,E}^-/p_2}{\Delta p_{2,E}^+/p_2}\left[\left(I_{2,\xi_f}^+\right)_{\tau_F}^{\tau_E} + \dots\right]\right\} \tag{26a}$$

$$\frac{\Delta p_2^-(x,t)}{p_2} = \left(I_{2,\xi_b}^-\right)_{\tau_B}^{t} + \left(\frac{\Delta p_{2,B}^-/p_2}{\Delta p_{2,B}^+/p_2}\right)\left\{\left(I_{2,\xi_d}^+\right)_{\tau_D}^{\tau_B} + \frac{\Delta p_{2,D}^+/p_2}{\Delta p_{2,D}^-/p_2}\left[\left(I_{2,\xi_g}^-\right)_{\tau_G}^{\tau_D} + \dots\right]\right\} \tag{26b}$$

The entropy perturbation can be found by evaluating equation (26a) at point C (so as to have $\Delta p_{2,C}^+$) and then utilizing equation (C10b).

Since the piston moves with constant velocity,[4] the reflection coefficient at the piston $((\Delta p_{2,A}^+/p_2)/(\Delta p_{2,A}^-/p_2)$, $(\Delta p_{2,D}^+/p_2)/(\Delta p_{2,D}^-/p_2)$, etc.) equals 1 (eq. (C5)). The reflection coefficient at the shock $((\Delta p_{2,B}^-/p_2)/(\Delta p_{2,B}^+/p_2)$, $(\Delta p_{2,E}^-/p_2)/(\Delta p_{2,E}^+/p_2)$, etc.) is zero for $M_s \equiv u_s/a_1 = 1$ (eq. (C10a)). For γ near 1.4, it is small for all values of M_s. In the latter cases, only the first few terms are required in equations (26). Figure 5(c) indicates the characteristic lines which are considered when the reflection coefficient at the shock is essentially zero.

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[4]If the piston velocity has small nonuniformities Δu_p, the problem can be treated as the superposition of the constant u_p case plus the case wherein pressure waves of magnitude $\Delta p_{2,A}^+/p_2 = \gamma\Delta u_{p,A}/a_2$, $\Delta p_{2,D}^+/p_2 = \gamma\Delta u_{p,D}/a_2$, etc. are generated at points A, D, F, ... The latter follow from the acoustic relations.

The piston-driven shock problem was treated in reference 6, from another viewpoint, to obtain the effect of small variations of piston velocity and the effect of a small linear area variation. The effect of an arbitrary area variation (e.g., wall boundary-layer effect), as well as arbitrary heat and momentum addition, can be treated by the present method.

The second example is the determination of nonuniformities in shock tubes due to unsteady wall boundary-layer action. This problem was treated in references 2 and 3, and the solution is summarized herein. Let regions 1 and 4 be the low- and high-pressure sections of a shock tube (fig. 6(a)). When the diaphragm breaks, a shock wave moves into region 1 with velocity u_s while an expansion wave (assumed to have negligible thickness) propagates into region 4 with velocity $-a_4$. A contact surface separates the shock compressed gas (region 2) from the expanded gas (region 3). Because of the fluid motion relative to the wall, a boundary layer develops along the wall between the expansion wave and the shock as indicated in figure 6(b). The problem is to find the perturbations (due to the boundary layer) at a typical point in region 2. The characteristics considered in references 2 and 3 are indicated in figure 6(c). The pressure perturbation at (x,t) is then found from (in the present notation)

$$\frac{\Delta p_2^{+}(x,t)}{p_2} = \left(I_{2,\xi_a}^{+}\right)_{\tau_A}^{t} + \frac{\Delta p_{2,A}^{+}/p_2}{\Delta p_{2,A}^{-}/p_2}\left(I_{2,\xi_e}^{-}\right)_{\tau_E}^{\tau_A} + \frac{\Delta p_{2,A}^{+}/p_2}{\Delta p_{3,A}^{+}/p_3}\left(I_{3,\xi_d}^{+}\right)_{\tau_D}^{\tau_A} \tag{27a}$$

$$\frac{\Delta p_2^{-}(x,t)}{p_2} = \left(I_{2,\xi_b}^{-}\right)_{\tau_B}^{t} + \frac{\Delta p_{2,B}^{-}/p_2}{\Delta p_{2,B}^{+}/p_2}\left[\left(I_{2,\xi_f}^{+}\right)_{\tau_F}^{\tau_B} + \frac{\Delta p_{2,F}^{+}/p_2}{\Delta p_{2,F}^{-}/p_2}\left(I_{2,\xi_h}^{-}\right)_{\tau_H}^{\tau_F} + \frac{\Delta p_{2,F}^{+}/p_2}{\Delta p_{3,F}^{+}/p_2}\left(I_{3,\xi_g}^{+}\right)_{\tau_G}^{\tau_F}\right] \tag{27b}$$

The bracketed term on the right side of equation (27b) equals $\Delta p_{2,B}^{+}/p_2$ and may be found by evaluating equation (27a) at point B. The entropy perturbation at (x,t) is found by evaluating equation (27a) at point C and applying equation (C10b). The source distribution in the integrals of equations (27) is given by

$$\left.\begin{aligned} \mu_2 &= \rho_2 l v_2/A \\ \mu_3 &= \rho_3 l v_3/A \end{aligned}\right\} \tag{28}$$

where v_2 and v_3 are the normal velocities (positive when directed inward) at the outer edge of the boundary layer in regions 2 and 3, respectively (as discussed in connection with eq. (2)). These velocities have the form

$$v_2(\xi,\tau) = \frac{K_2}{(u_s\tau - \xi)^{N_2}} \tag{29a}$$

$$v_3(\xi,\tau) = \frac{K_3}{(a_4\tau + \xi)^{N_3}} \tag{29b}$$

where K_2 and K_3 are constants and $N = 1/2$ or $1/5$ for wholly laminar or wholly turbulent boundary layers, respectively (refs. 2, 9, and 10).[5] By using these forms, the integrals in equations (27a) and (27b) are readily evaluated. A discussion of the resulting shock-tube nonuniformities is given in references 2 and 3.

The linearized characteristic method developed in reference 4 is an alternate method for determining shock-tube nonuniformities (see discussion following eqs. (18)). However, the source distribution in reference 4 was based, in effect, on $f(\xi,\tau)$ (obtained by averaging the boundary-layer wall shear across the tube cross section) and $q(\xi,\tau)$ (obtained by averaging the heat transfer at the wall and the dissipation within the boundary layer) as opposed to the use herein of $\mu(\xi,\tau)$ (i.e., effective area change due to boundary layer). Since $f(\xi,\tau)$ introduces an antisymmetric wave pattern (with respect to sign), the results of reference 4 differ by more than just a factor of proportionality from the results of references 2 and 3 (particularly with regard to the perturbations near the contact surface). This point is discussed further in references 2 and 3. For the case of relatively thin wall boundary layers, it is clear that the source distribution used in references 2 and 3 is the correct one. The viewpoint of reference 4 may have some validity, however, when the boundary layer spans the entire tube cross section (provided the proper boundary-layer theory is used to obtain the wall shear, heat transfer, and dissipation and provided the averaged perturbations are sufficiently small to justify a linearized approach). The authors of reference 4 have extended their work, and the results are given in reference 11.

The flow resulting when a weak deflagration is initiated at the closed end of a tube is another example of a basic flow containing

[5] The expression for $v_3(\xi,\tau)$ is somewhat in error because of neglect of the finite width of the expansion wave and because of the presence of gas, originally from region 2, at the wall between the diaphragm location ($x = 0$) and the contact surface (see ref. 10). Use of eq. (29b) seems adequate for most purposes (except, possibly, for strong shocks).

centered discontinuities (see p. 225 of ref. 12). The basic flow consists of a shock wave, followed by a planar flame front, both propagating with constant speed (fig. 7). The shock induces a velocity u_2 in region 2. The closed-end boundary condition requires $u_3 = 0$. (For a given flame, the shock strength is determined by the boundary condition $u_3 = 0$.) The effect of the wall boundary layer on the flame and shock propagation can be treated in a manner similar to the shock-tube problem. Thus, the source distribution is taken to be $\mu = \rho l v/A$, where v is found from boundary-layer theory. Note that the boundary layer in region 2 is the same as that in region 2 of the shock tube. However, since $u_3 = 0$ in the present example, the boundary layer in region 3 is essentially a thermal boundary layer only. The solution of this thermal boundary layer can be found using the methods of reference 9 provided the flow across the flame front is assumed uniform (i.e., the boundary layer from region 2 is neglected). The characteristics contributing to the perturbations at (x,t) in region 2 are indicated in figure 7(c). It is assumed therein that the reflection coefficient at the shock is zero (appropriate for weak shocks). Transfer functions, for use at a planar flame front, are derived in appendix C. The effect of flame front distortion and the corresponding changes in the mean flame speed are not considered in the present formulation. The effect of temperature and turbulence level on flame speed is also neglected.

CONCLUDING REMARKS

A source distribution method has been presented for determining perturbations due to small unsteady area variations, mass, momentum, and heat additions in a uniform (or piecewise uniform) basic one-dimensional flow.

In the present method, the perturbed flow field is decomposed into three mutually independent wave systems, namely (1) upstream propagating acoustic waves Δp^-, (2) downstream propagating acoustic waves Δp^+, and (3) entropy waves Δs. The pressure and velocity perturbations at a given section depend only on the local values of Δp^- and Δp^+ and are independent of the local value of Δs. However, a knowledge of Δs is required if the local density and temperature perturbations are to be found. The present method is compared with the linearized method of characteristics (e.g., ref. 4) in the paragraph following equations (18). The linearized method of characteristics employs ΔP, ΔQ, and Δs as the dependent variables. But ΔP and ΔQ are not independent of Δs for nonisentropic flows. Hence, ΔP, ΔQ, and Δs do not form a system of three mutually independent waves as do Δp^+, Δp^-, and Δs.

Several alternate viewpoints can be used to solve the problems considered in the body of the report (e.g., refs. 4, 6, 7, and 8). It is hoped that the present elementary discussion adds additional physical insight for these flows.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, February 6, 1959

APPENDIX A

SYMBOLS

A	cross-sectional area of tube
$A,B,C,\ldots$	points on characteristic lines
a	speed of sound
$a,b,c,\ldots$	characteristic lines
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
f	body force, per unit volume, acting in +x direction
I	see eqs. (23a) and (23b)
l	perimeter of tube cross section
M	Mach number, u/a
m	mass addition, per unit volume, per unit time
P,Q	Riemann variables (eqs. (13d) and (13e))
p	pressure
q	heat addition, per unit volume, per unit time
R	gas constant
s	entropy
T	temperature
t	time
u	velocity in +x direction
v	velocity (normal to wall) of fluid particle at wall, or at outer edge of thin boundary layer (positive inward)
w	velocity of discontinuity

x	distance along tube, taken in stream direction
γ	ratio of specific heats
Δ	prefix denoting perturbation
λ	$(c_pT)_2/(c_pT)_1$
μ	equivalent mass addition, per unit volume, per unit time
ξ	distance along tube, defining source point
ρ	density
τ	time, defining source point

Subscripts:

1,2,3,...	regions
A,B,C,...	points on characteristic lines
f	flame front
s	shock wave

Superscripts:

$(\ )^+$	perturbation associated with acoustic wave propagating in +x direction
$(\ )^-$	perturbation associated with acoustic wave propagating in -x direction
$\overline{(\ )}$	used with μ, q, and f to indicate source strength defined on a per unit cross-sectional area basis

Special Notation:

$\frac{\delta_{\pm}(\ )}{\delta t}$	eq. (13f)
$\frac{D(\ )}{Dt}$	eq. (13g)

$$\left.\begin{matrix} I^{+}_{2,\xi_a},\ldots \\ \\ I^{-}_{1,\xi_e},\ldots \end{matrix}\right\}$$ eqs. (23a) and (23b)

Example:

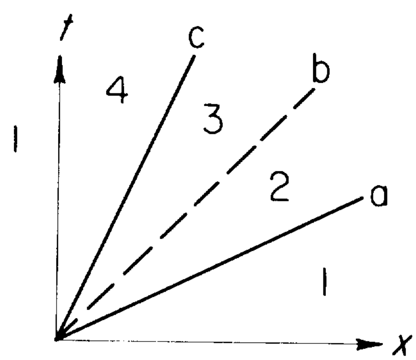
$\Delta p^{+}_{2,A}$ pressure perturbation at point A, region 2, associated with acoustic wave propagating in +x direction

APPENDIX B

ELEMENTAL SOURCE SOLUTION FOR $M > 1$

The wave due to an elemental source in a subsonic stream was derived in the body of the report. The case of an elemental source in a supersonic stream is treated herein.

Planar mass, momentum, and heat sources are placed at $x = 0$ at $t = 0$ and remain at constant strength thereafter. The resulting t-x diagram, for $M > 1$, is shown in sketch (h). Line a is a downstream propagating acoustic wave, line b is a contact surface, line c is an upstream propagating acoustic wave (which is swept downstream, since $M > 1$), and line $x = 0$ is a line across which discontinuities also originate. The perturbations are constant in each of regions 2 to 4. These are found by considering the perturbations arising from each of the above-mentioned lines.



(h)

Line a: The acoustic relations give

$$\left.\begin{aligned} \Delta p_2 &= \rho a \Delta u_2 \\ \Delta \rho_2 &= \frac{\rho}{a} \Delta u_2 \end{aligned}\right\} \tag{B1}$$

Line b: Since this line is a contact surface,

$$\left.\begin{aligned} \Delta p_3 &= \Delta p_2 \\ \Delta u_3 &= \Delta u_2 \end{aligned}\right\} \tag{B2}$$

Line c: From acoustic relations,

$$\left.\begin{aligned} \Delta p_3 - \Delta p_4 &= -\rho a(\Delta u_3 - \Delta u_4) \\ \Delta \rho_3 - \Delta \rho_4 &= -\frac{\rho}{a}(\Delta u_3 - \Delta u_4) \end{aligned}\right\} \tag{B3}$$

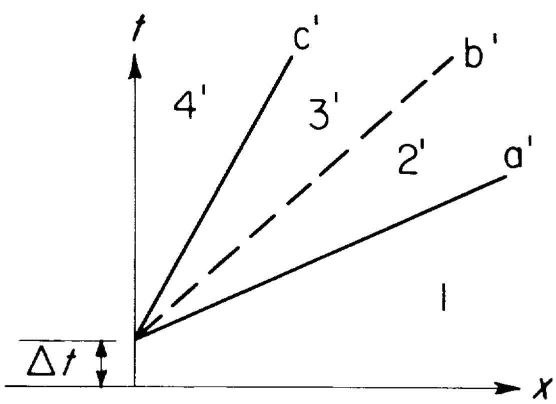
<u>Line $x = 0$</u>: Integration of equations (1) across $x = 0$ gives

$$\left.\begin{aligned} u\Delta\rho_4 + \rho\Delta u_4 &= \bar{\mu} \\ u\Delta u_4 + (1/\rho)\Delta p_4 &= \bar{f}/\rho \\ u\left(\frac{\Delta p_4}{p} - \gamma\,\frac{\Delta\rho_4}{\rho}\right) &= \frac{\bar{q}}{c_v\rho T} \end{aligned}\right\} \tag{B4}$$

Equations (B1) to (B4) are nine equations in nine unknowns. The solutions can be written

$$\left.\begin{aligned} \frac{\Delta p_2}{p} = \frac{\Delta p_3}{p} &= \frac{\gamma}{2\rho a(1 + M)}\left(\bar{\mu} + \frac{\bar{q}}{c_pT} + \frac{\bar{f}}{a}\right) \\ \frac{\Delta p_4}{p} &= \frac{\gamma M}{\rho a(M^2 - 1)}\left(\bar{\mu} + \frac{\bar{q}}{c_pT} - \frac{\bar{f}}{Ma}\right) \\ \frac{\Delta s_3}{c_v} = \frac{\Delta s_4}{c_v} &= \frac{\bar{q}}{c_v\rho uT} \end{aligned}\right\} \tag{B5}$$

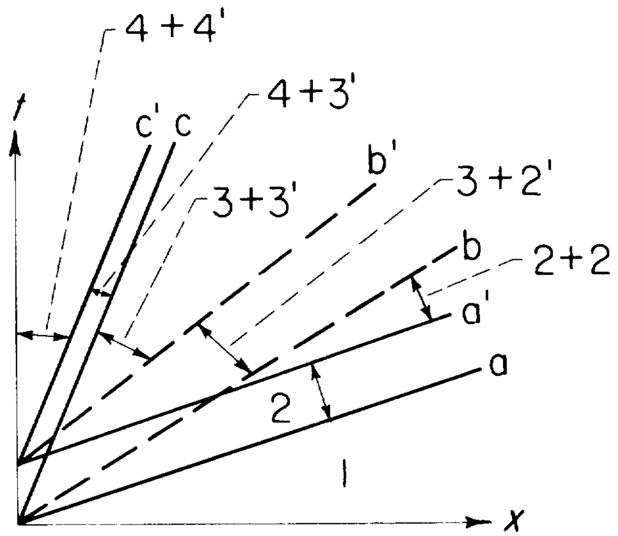
The other perturbations can then be found from equations (B1) to (B4).



(i)

Now assume that equal and opposite (with respect to the previous example) planar mass momentum and heat sources are placed at $x = 0$ at $t = \Delta t$ and remain at constant strength thereafter. The t-x diagram is given in sketch (i). The lines are displaced upward by an amount Δt from those in sketch (h). The perturbations in regions 2', 3', and 4' are equal and opposite to those in regions 2, 3, and 4.

The flow field associated with an elemental mass, momentum, and heat source at $x = 0$, $t = 0$ is found by linearly superposing the flows in sketches (h) and (i). The superposition is indicated in sketch (j).



(j)

The perturbations are zero everywhere except between lines a-a', b-b', and c-c' (since the perturbations in regions 2, 3, and 4 are equal and opposite to those in regions 2', 3', and 4', respectively). The perturbations between lines a-a' can be written

$$\frac{\Delta p^+}{p} = \frac{\gamma}{2\rho a(1 + M)}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} + \frac{\bar{f}}{a}\right) \tag{B6a}$$

$$\frac{\Delta u^+}{a} = \frac{\Delta \rho^+}{\rho} = \frac{1}{\gamma}\,\frac{\Delta p^+}{p}$$

Between lines c-c', the net perturbations are

$$\frac{\Delta p^-}{p} = \frac{\gamma}{2\rho a(M - 1)}\left(\bar{\mu} + \frac{\bar{q}}{c_p T} - \frac{\bar{f}}{a}\right) \tag{B6b}$$

$$-\frac{\Delta u^-}{a} = \frac{\Delta \rho^-}{\rho} = \frac{1}{\gamma}\,\frac{\Delta p^-}{p}$$

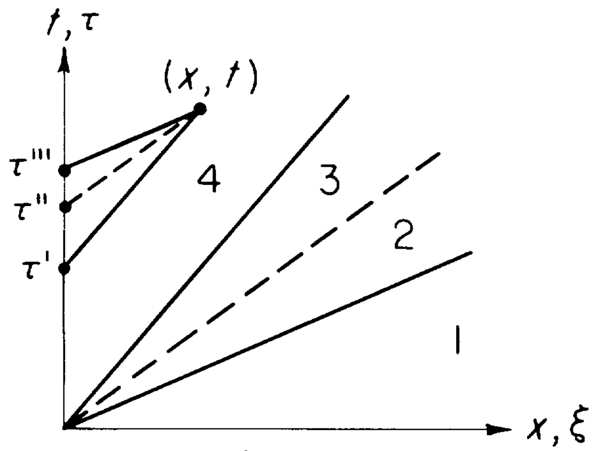
Between lines b-b', the net perturbations are $\Delta p = \Delta u = 0$ and

$$\frac{\Delta s}{c_v} = \frac{\bar{q}}{c_v \rho u T} \tag{B6c}$$

$$\frac{\Delta \rho}{\rho} = -\frac{\Delta s}{c_p}$$

By considering the limit as $\Delta t \to 0$, equations (B6) define the perturbations due to an elemental planar source at $x = 0$, $t = 0$ in a supersonic stream. These equations differ from those for the $M < 1$ case (i.e., eqs. (9)) only in the terms $(1 - M)$, $(M - 1)$ appearing in the expressions for Δp^-. Use of $|1 - M|$ in equations (10) makes the latter applicable for both $M < 1$ and $M > 1$.

It may be of interest to interpret the flow in sketch (h) in terms of the elemental source solutions. For example, the net pressure perturbation at an arbitrary point in region 4 (see sketch (k)) may be interpreted as consisting of $\Delta p = \Delta p^+ + \Delta p^-$ where Δp^+ is the downstream acoustic wave originating from the source at $\xi = 0$, $\tau''' = t - [x/(u + a)]$ while Δp^- is the upstream propagating acoustic wave originating from the source at $\xi = 0$, $\tau' = t - [x/(u - a)]$. The entropy perturbation at (x,t) is equal to the entropy perturbation which originated from the source at $\xi = 0$, $\tau'' = t - (x/u)$. The greater generality of the elemental source viewpoint is illustrated by considering the source strength at $\xi = 0$ to vary with time. Then, the perturbations at any point can be found directly from equations (B6) provided the right sides are evaluated at τ''', τ', and τ'', respectively.



(k)

APPENDIX C

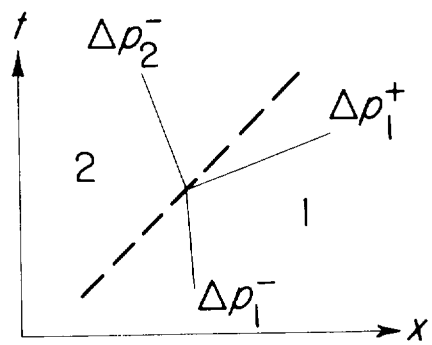
TRANSFER FUNCTIONS

In the body of the report it was necessary to know the ratio of the generated wave to the incident wave when an acoustic wave, or entropy wave, impinged on a discontinuity. Such ratios were termed "transfer functions." The transfer functions for various discontinuities and incident waves are found herein.

Contact Surface

The requirement that pressure and velocity be continuous across a contact surface leads to (e.g., ref. 2):

<u>Incident wave Δp_1^-:</u>

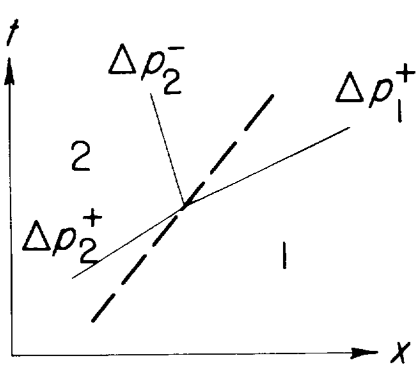


(l)

$$\frac{\Delta p_1^+/p_1}{\Delta p_1^-/p_1} = \frac{\dfrac{\gamma_2 a_1}{\gamma_1 a_2} - 1}{\dfrac{\gamma_2 a_1}{\gamma_1 a_2} + 1} \tag{C1}$$

$$\frac{\Delta p_2^-/p_2}{\Delta p_1^-/p_1} = \frac{2}{\dfrac{\gamma_1 a_2}{\gamma_2 a_1} + 1} \tag{C2}$$

<u>Incident wave Δp_2^+:</u>



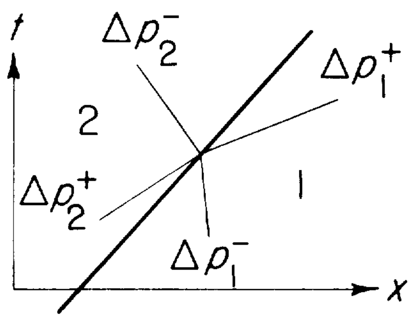
(m)

$$\frac{\Delta p_1^+/p_1}{\Delta p_2^+/p_2} = \frac{2}{\dfrac{\gamma_2 a_1}{\gamma_1 a_2} + 1} \tag{C3}$$

$$\frac{\Delta p_2^-/p_2}{\Delta p_2^+/p_2} = \frac{\dfrac{\gamma_1 a_2}{\gamma_2 a_1} - 1}{\dfrac{\gamma_1 a_2}{\gamma_2 a_1} + 1} \tag{C4}$$

Constant-Velocity Discontinuity

In this case the velocity of the discontinuity is assumed to be unperturbed. A piston moving at constant velocity is an example.

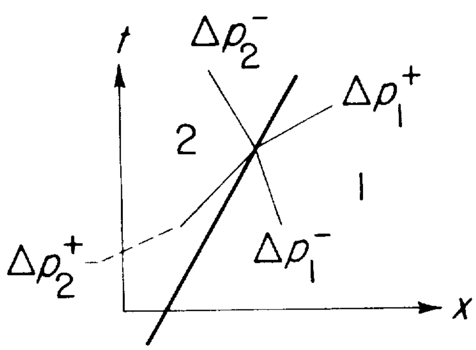


(n)

$$\frac{\Delta p_1^+/p_1}{\Delta p_1^-/p_1} = \frac{\Delta p_2^-/p_2}{\Delta p_2^+/p_2} = 1 \qquad (C5)$$

Constant-Pressure Discontinuity

In this case, the pressure at the discontinuity is assumed to be unperturbed. The open end of a tube is an example.

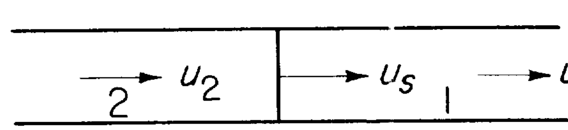


(o)

$$\frac{\Delta p_1^+/p_1}{\Delta p_1^-/p_1} = \frac{\Delta p_2^-/p_2}{\Delta p_2^+/p_2} = -1 \qquad (C6)$$

Shock-Wave Discontinuity

The derivation of the transfer functions for this case will be outlined. Let $M_s \equiv (u_s - u_1)/a_1$ designate the shock Mach number relative to the fluid in region 1.



(p)

Normal shock relations give

$$\frac{p_2}{p_1} = \frac{2\gamma M_s^2 - (\gamma - 1)}{\gamma + 1} \tag{C7a}$$

$$\frac{u_2 - u_1}{a_1} = \frac{2}{\gamma + 1} \frac{M_s^2 - 1}{M_s} \tag{C7b}$$

$$\frac{s_2 - s_1}{c_v} = \ln\left\{\left[\frac{2\gamma M_s^2 - (\gamma - 1)}{\gamma + 1}\right] \bigg/ \left[\frac{(\gamma + 1)M_s^2}{(\gamma - 1)M_s^2 + 2}\right]^\gamma\right\} \tag{C7c}$$

Taking the differential of equations (C7a) and (C7b), eliminating ΔM_s, and noting $\frac{\Delta a_1}{a_1} = \frac{1}{2}\left(\frac{\gamma - 1}{\gamma} \frac{\Delta p_1}{p_1} + \frac{1}{\gamma} \frac{\Delta s_1}{c_v}\right)$ yield

$$\frac{\Delta p_2}{p_2} - \frac{\Delta p_1}{p_1} = z\left[\frac{a_2}{a_1}\left(\gamma \frac{\Delta u_2}{a_2}\right) - \gamma \frac{\Delta u_1}{a_1} - \frac{\gamma - 1}{\gamma + 1} \frac{M_s^2 - 1}{M_s}\left(\frac{\Delta p_1}{p_1} + \frac{1}{\gamma - 1} \frac{\Delta s_1}{c_v}\right)\right] \tag{C8}$$

where

$$z = \frac{2(\gamma + 1)M_s^3}{\left[2\gamma M_s^2 - (\gamma - 1)\right]\left(M_s^2 + 1\right)}$$

The differential of equation (C7c), upon elimination of ΔM_s, yields

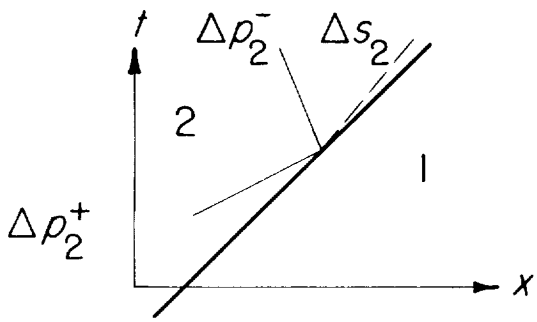
$$\frac{\Delta s_2 - \Delta s_1}{c_v} = \left(\frac{\Delta p_2}{p_2} - \frac{\Delta p_1}{p_1}\right) y \tag{C9}$$

where

$$y = \frac{(\gamma - 1)(M_s^2 - 1)^2}{M_s^2\left[(\gamma - 1)M_s^2 + 2\right]}$$

Use of the acoustic relations $\gamma\Delta u^+/a = \Delta p^+/p$ and $\gamma\Delta u^-/a = -\Delta p^-/p$ in equations (C8) and (C9) then yields the following transfer functions:

<u>Incident wave Δp_2^+:</u>

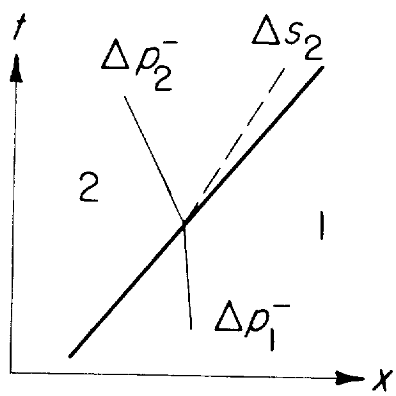


(q)

$$\frac{\Delta p_2^-/p_2}{\Delta p_2^+/p_2} = \frac{z\,\dfrac{a_2}{a_1} - 1}{z\,\dfrac{a_2}{a_1} + 1} \qquad \text{(C10a)}$$

$$\frac{\Delta s_2/c_v}{\Delta p_2^+/p_2} = \left(\frac{\Delta p_2^-/p_2}{\Delta p_2^+/p_2} + 1\right) y \qquad \text{(C10b)}$$

<u>Incident wave Δp_1^-:</u>

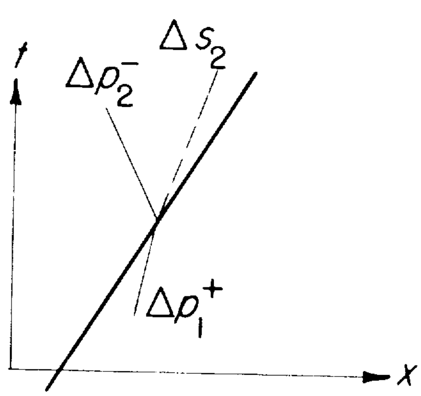


(r)

$$\frac{\Delta p_2^-/p_2}{\Delta p_1^-/p_1} = \frac{1 + z\left(1 - \dfrac{\gamma - 1}{\gamma + 1}\,\dfrac{M_s^2 - 1}{M_s}\right)}{1 + z\,\dfrac{a_2}{a_1}} \qquad \text{(C11a)}$$

$$\frac{\Delta s_2/c_v}{\Delta p_1^-/p_1} = \left(\frac{\Delta p_2^-/p_2}{\Delta p_1^-/p_1} - 1\right) y \qquad \text{(C11b)}$$

<u>Incident wave Δp_1^+:</u>

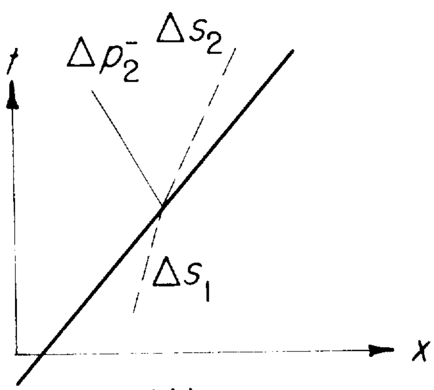


(s)

$$\frac{\Delta p_2^-/p_2}{\Delta p_1^+/p_1} = \frac{1 - z\left(1 + \dfrac{\gamma - 1}{\gamma + 1}\,\dfrac{M_s^2 - 1}{M_s}\right)}{1 + z\,\dfrac{a_2}{a_1}} \qquad \text{(C12a)}$$

$$\frac{\Delta s_2/c_v}{\Delta p_1^+/p_1} = \left(\frac{\Delta p_2^-/p_2}{\Delta p_1^+/p_1} - 1\right) y \qquad \text{(C12b)}$$

Incident wave $\Delta s_1/c_v$:



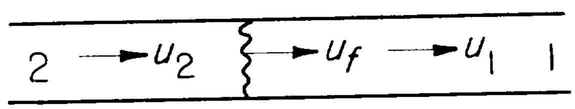
(t)

$$\frac{\Delta p_2^-/p_2}{\Delta s_1/c_v} = -\frac{M_s^2 - 1}{(\gamma + 1)M_s} \frac{z}{1 + z\dfrac{a_2}{a_1}} \qquad \text{(C13a)}$$

$$\frac{\Delta s_2/c_v}{\Delta s_1/c_v} = 1 + \left(\frac{\Delta p_2^-/p_2}{\Delta s_1/c_v}\right)y \qquad \text{(C13b)}$$

Planar-Flame-Front Discontinuity

Transfer functions for a planar flame front are derived herein by assuming:



(u)

(1) $\left[(u_f - u_1)/a_1\right]^2 \ll 1$; $\left[(u_f - u_2)/a_2\right]^2 \ll 1$

(2) The flame speed relative to the gas in region 1, $u_f - u_1$, is constant

(3) The heat release at the flame per unit mass, per unit cross-sectional area (denoted $\tilde{Q}$), is constant.

The equations of motion, relative to the flame, are

Continuity: $$\rho_1(u_f - u_1) = \rho_2(u_f - u_2) \qquad \text{(C14a)}$$

Momentum: $$p_1 = p_2 \qquad \text{(C14b)}$$

Energy: $$(c_pT)_1 + \tilde{Q} = (c_pT)_2 \qquad \text{(C14c)}$$

State: $$p_1 = \rho_1R_1T_1 \qquad p_2 = \rho_2R_2T_2 \qquad \text{(C14d)}$$

Assumption (1) is incorporated in equations (C14b) and (C14c). The differential of these equations yields, respectively (noting assumptions (2), i.e., $\Delta(u_f - u_1) = 0$ or $\Delta u_f = \Delta u_1$, and (3), i.e., $\Delta Q = 0$),

$$\frac{\Delta\rho_1}{\rho_1} - \frac{\Delta\rho_2}{\rho_2} = \frac{a_1}{u_f - u_2}\frac{1}{\gamma_1}\left(\gamma_1 \frac{\Delta u_1}{a_1} - \frac{a_2\gamma_1}{a_1\gamma_2}\gamma_2\frac{\Delta u_2}{a_2}\right) \tag{C15a}$$

$$\frac{\Delta p_1}{p_1} = \frac{\Delta p_2}{p_2} \tag{C15b}$$

$$\frac{\Delta T_1}{T_1} = \lambda \frac{\Delta T_2}{T_2} \tag{C15c}$$

$$\frac{\Delta\rho_2}{\rho_2} - \frac{\Delta\rho_1}{\rho_1} = \left(\frac{\lambda - 1}{\lambda}\right)\frac{\Delta T_1}{T_1} \tag{C15d}$$

where $\lambda = (c_pT)_2/(c_pT)_1$. Eliminating the density terms between equations (C15a) and (C15d) yields

$$\frac{\gamma_1 Z}{\gamma_1 - 1}\frac{\Delta T_1}{T_1} = \frac{a_2}{a_1}\frac{\gamma_1}{\gamma_2}\gamma_2\frac{\Delta u_2}{a_2} - \gamma_1\frac{\Delta u_1}{a_1} \tag{C16}$$

where

$$Z = \frac{\gamma_1}{\gamma_2}(\gamma_2 - 1)(\lambda - 1)\left(\frac{u_f - u_1}{a_1}\right)$$

But $\Delta T_1/T_1 = \left[(\gamma_1 - 1)/\gamma_1\right](\Delta p_1/p_1) + (1/\gamma_1)(\Delta s_1/c_{v,1})$, so that equation (C16) can be written

$$Z\left(\frac{\Delta p_1}{p_1} + \frac{1}{\gamma_1 - 1}\frac{\Delta s_1}{c_{v,1}}\right) = \frac{a_2\gamma_1}{a_1\gamma_2}\gamma_2\frac{\Delta u_2}{a_2} - \gamma_1\frac{\Delta u_1}{a_1} \tag{C17}$$

Also, from the definition of entropy,

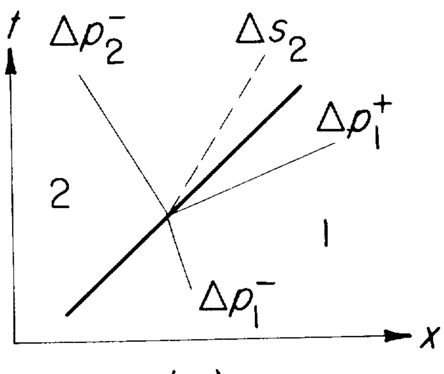
$$\begin{aligned}\frac{\Delta s_2}{c_{v,2}} &= \frac{\Delta p_2}{p_2} - \gamma_2\frac{\Delta\rho_2}{\rho_2} = (1 - \gamma_2)\frac{\Delta p_2}{p_2} + \gamma_2\frac{\Delta T_2}{T_2} \\ &= (1 - \gamma_2)\frac{\Delta p_1}{p_1} + \frac{\gamma_2}{\lambda}\frac{\Delta T_1}{T_1} \\ &= Y\frac{\Delta p_1}{p_1} + \frac{1}{\lambda}\frac{\gamma_2}{\gamma_1}\frac{\Delta s_1}{c_{v,1}}\end{aligned} \tag{C18}$$

where

$$Y = (\gamma_1 - 1)\left[\frac{1}{\lambda}\frac{\gamma_2}{\gamma_1} - \frac{(\gamma_2 - 1)}{(\gamma_1 - 1)}\right]$$

Equations (C15b), (C17), and (C18) together with the acoustic relations then yield the following transfer functions:

Incident wave Δp_1^-:



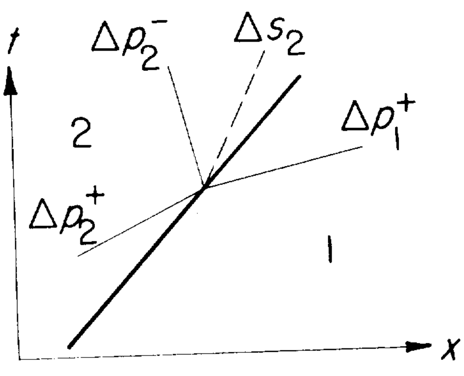
(v)

$$\frac{\Delta p_1^+/p_1}{\Delta p_1^-/p_1} = \frac{1 - Z - \frac{\gamma_1}{\gamma_2}\frac{a_2}{a_1}}{1 + Z + \frac{\gamma_1}{\gamma_2}\frac{a_2}{a_1}} \qquad \text{(C19a)}$$

$$\frac{\Delta p_2^-/p_2}{\Delta p_1^-/p_1} = 1 + \frac{\Delta p_1^+/p_1}{\Delta p_1^-/p_1} \qquad \text{(C19b)}$$

$$\frac{\Delta s_2/c_{v,2}}{\Delta p_1^-/p_1} = Y\,\frac{\Delta p_2^-/p_2}{\Delta p_1^-/p_1} \qquad \text{(C19c)}$$

Incident wave Δp_2^+:



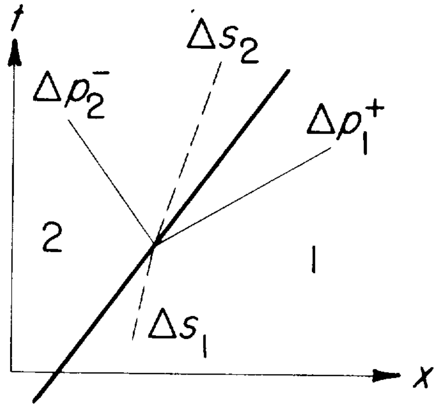
(w)

$$\frac{\Delta p_2^-/p_2}{\Delta p_2^+/p_2} = -\,\frac{1 + Z - \frac{\gamma_1}{\gamma_2}\frac{a_2}{a_1}}{1 + Z + \frac{\gamma_1}{\gamma_2}\frac{a_2}{a_1}} \qquad \text{(C20a)}$$

$$\frac{\Delta p_1^+/p_1}{\Delta p_2^+/p_2} = 1 + \frac{\Delta p_2^-/p_2}{\Delta p_2^+/p_2} \qquad \text{(C20b)}$$

$$\frac{\Delta s_2/c_{v,2}}{\Delta p_2^+/p_2} = Y\,\frac{\Delta p_1^+/p_1}{\Delta p_2^+/p_2} \qquad \text{(C20c)}$$

Incident wave Δs_1:



(x)

$$\frac{\Delta p_1^+/p_1}{\Delta s_1/c_{v,1}} = \frac{\Delta p_2^-/p_2}{\Delta s_1/c_{v,1}}$$

$$= \frac{-1}{\gamma_1 - 1} \frac{Z}{1 + Z + \frac{\gamma_1}{\gamma_2} \frac{a_2}{a_1}} \qquad \text{(C21a)}$$

$$\frac{\Delta s_2/c_{v,2}}{\Delta s_1/c_{v,1}} = \frac{1}{\lambda} \frac{\gamma_2}{\gamma_1} + Y \frac{\Delta p_1^+/p_1}{\Delta s_1/c_{v,1}} \qquad \text{(C21b)}$$

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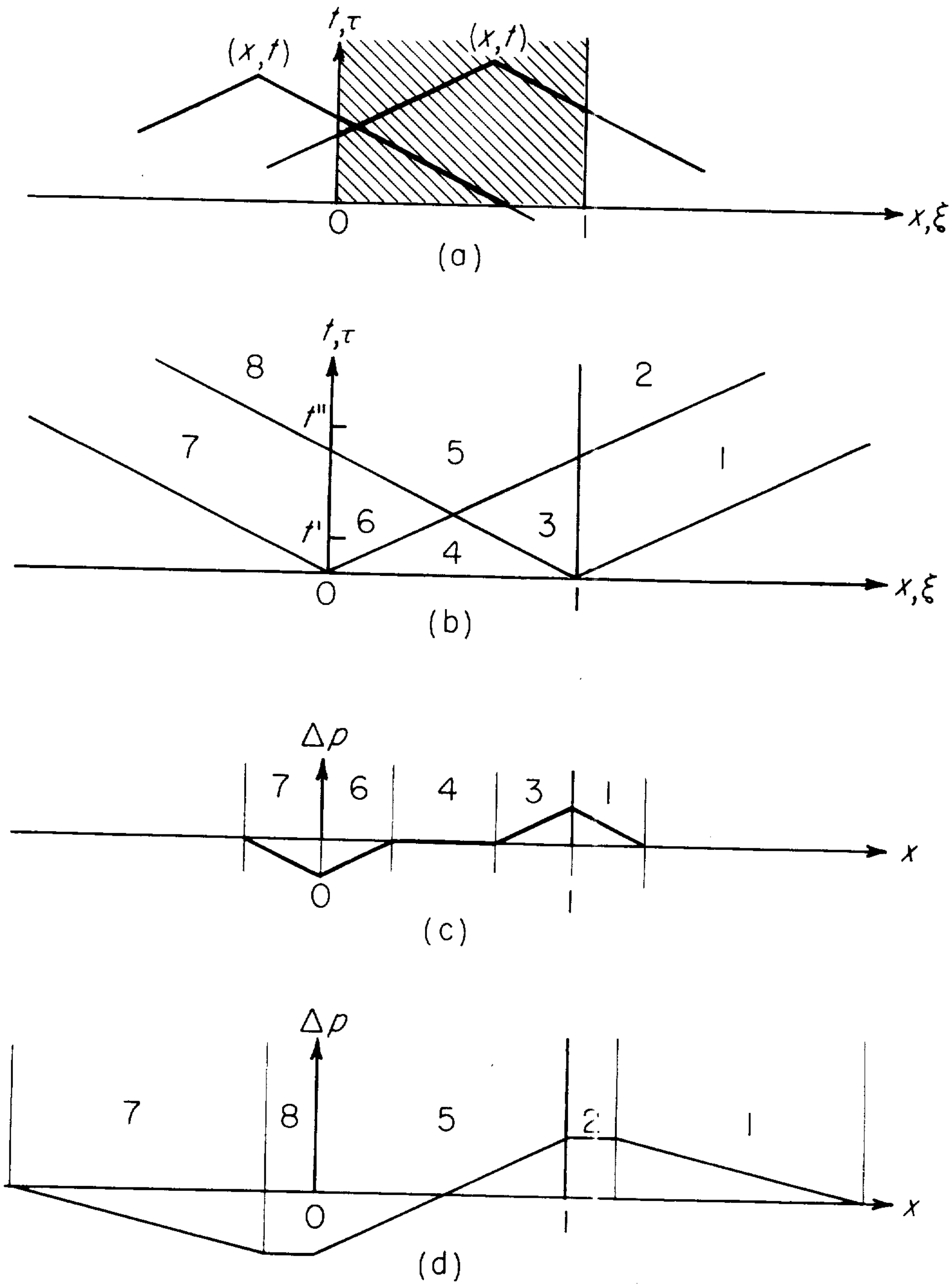
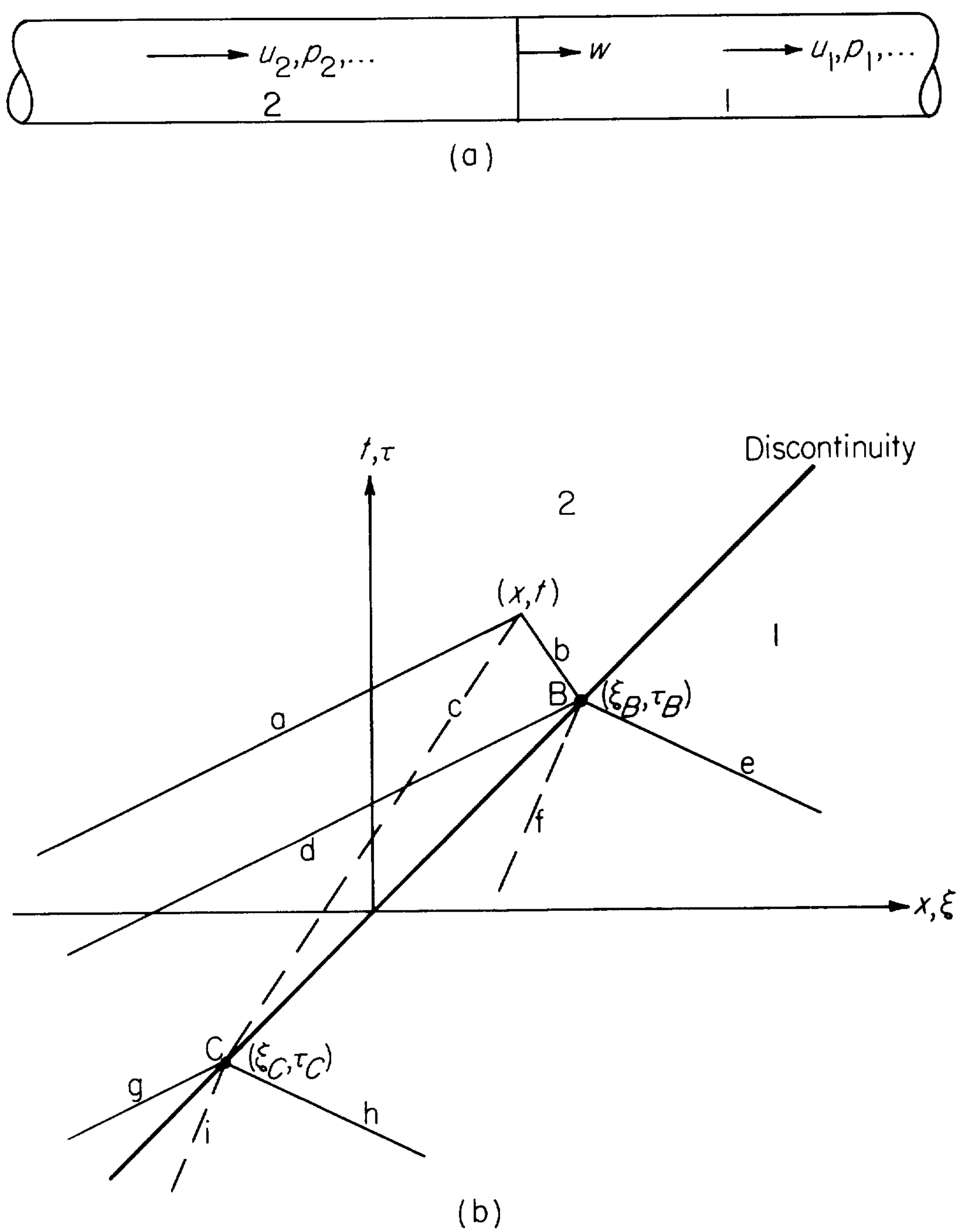
(a) Area variation.
(b) Sources affecting point (x,t) for subsonic flow.
(c) Sources affecting point (x,t) for supersonic flow.

Figure 2. - Source distribution method for finding steady



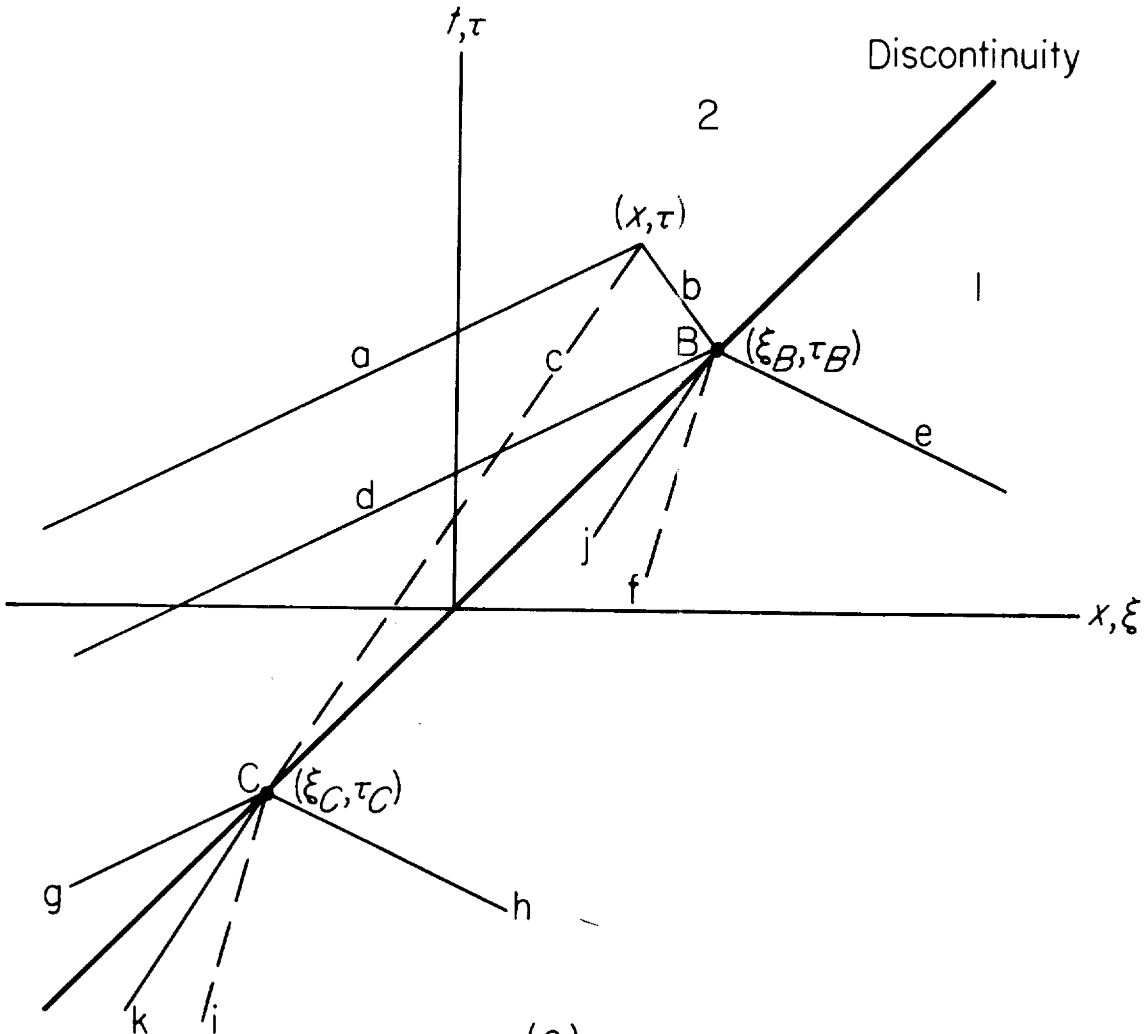
(a) Volumetric body forces ($0 \leq \xi \leq 1$, $\tau \geq 0$).
(b) Regions having different expressions for local pressure.
(c) Pressure distribution at time $t = t'$.
(d) Pressure distribution at time $t = t''$.

Figure 3. - Perturbations induced by volumetric body force sources in uniform tube containing stationary gas. Sources uniform and nonzero for $0 \leq \xi \leq 1$, $\tau \geq 0$.



(a) Basic flow containing discontinuity.
(b) Characteristics contributing to perturbations at (x,t) for case $0 < M_1 < 1$, $0 < M_2 < 1$, $u_1 < w < u_1 + a_1$, $u_2 < w$. Since $(w - u_1)/a_1 < 1$, discontinuity moves with subsonic velocity relative to gas in region 1.

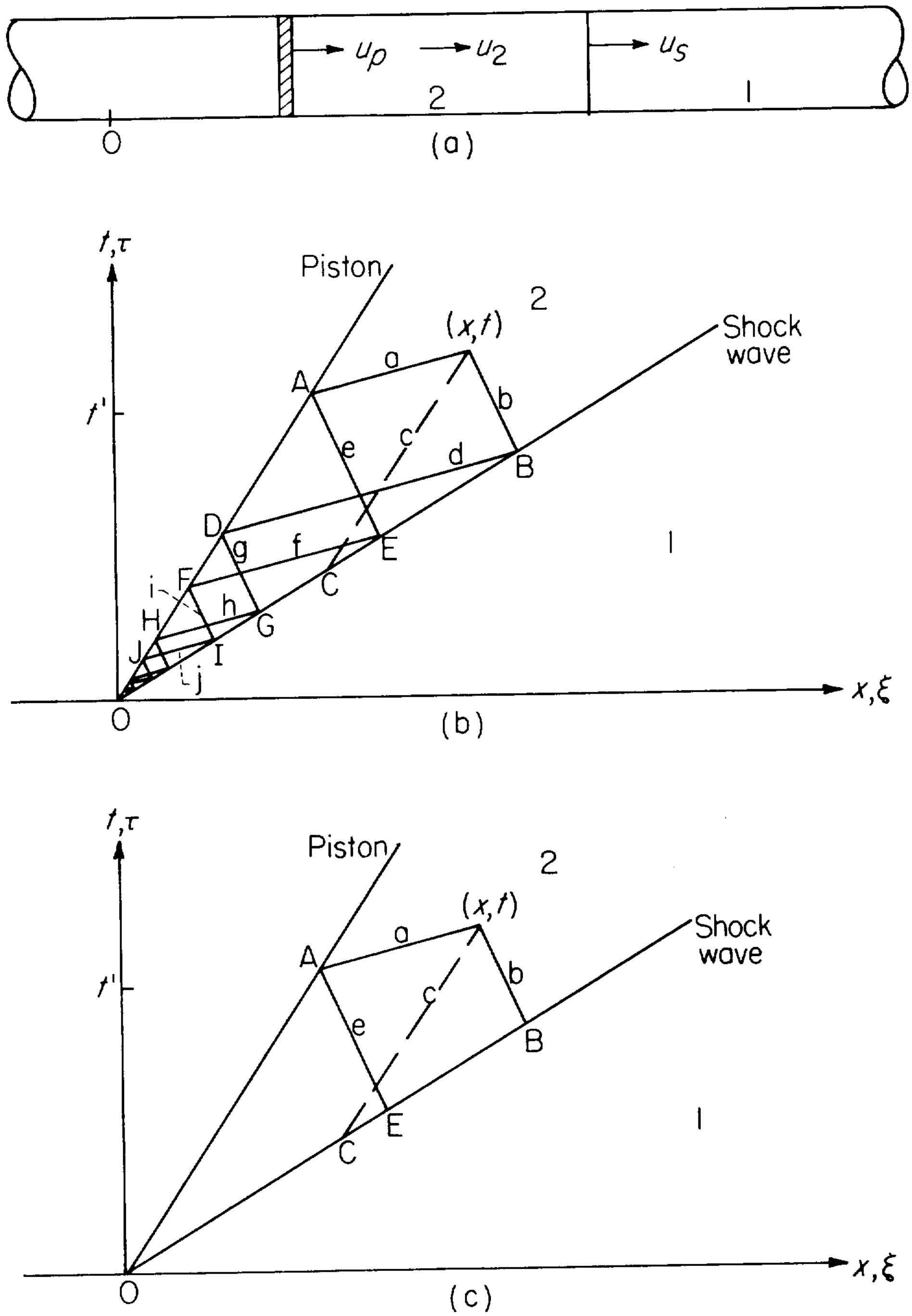
Figure 4. - Characteristics for finding perturbations in flows containing a discontinuity.



(c)

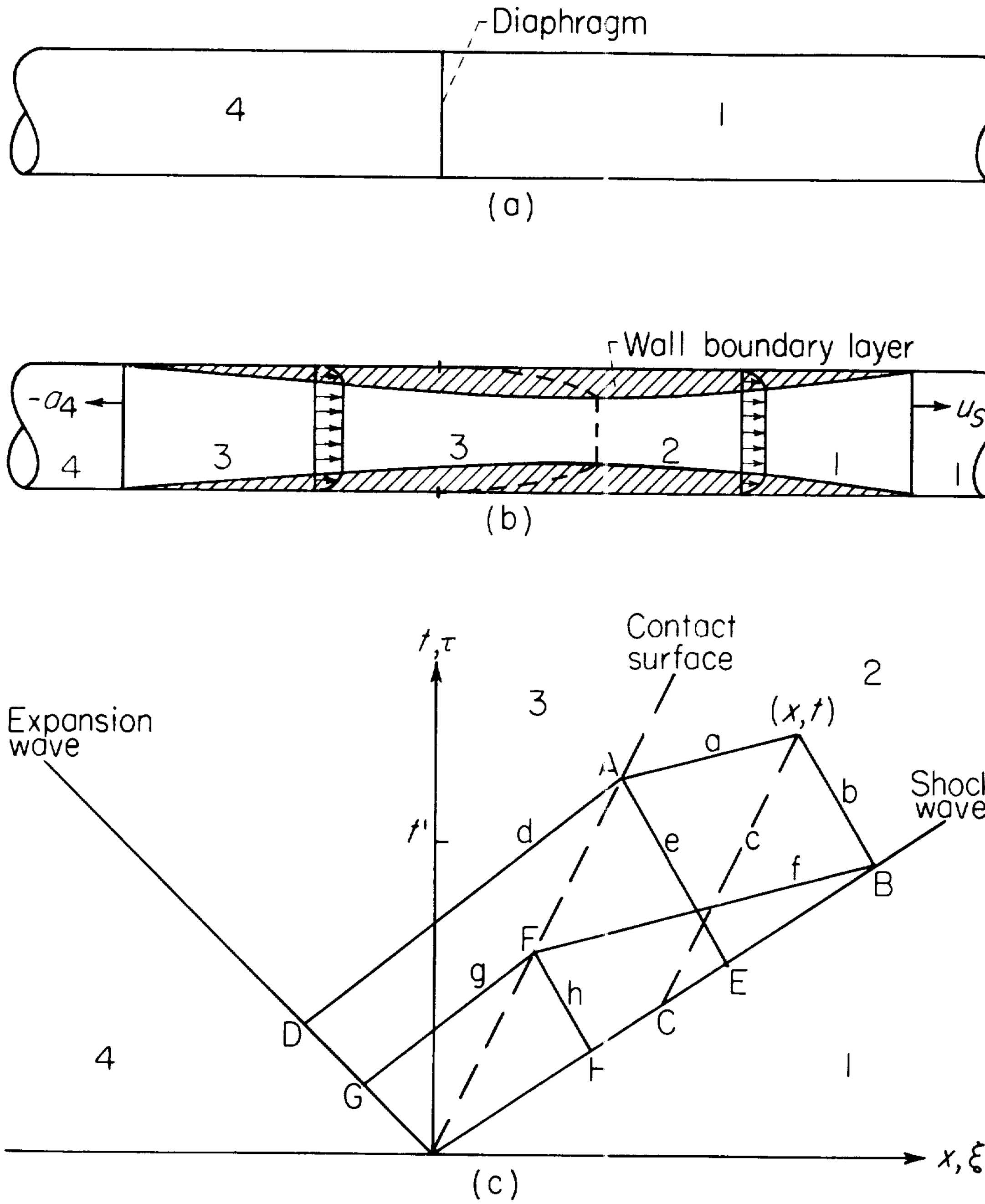
(c) Characteristics contributing to perturbations at (x,t) for case $0 < M_1 < 1$, $0 < M_2 < 1$, $u_1 + a_1 < w$, $u_2 < w$. Since $(w - u_1)/a_1 > 1$, discontinuity moves with supersonic velocity relative to gas in region 1.

Figure 4. - Concluded. Characteristics for finding perturbations in flows containing a discontinuity.



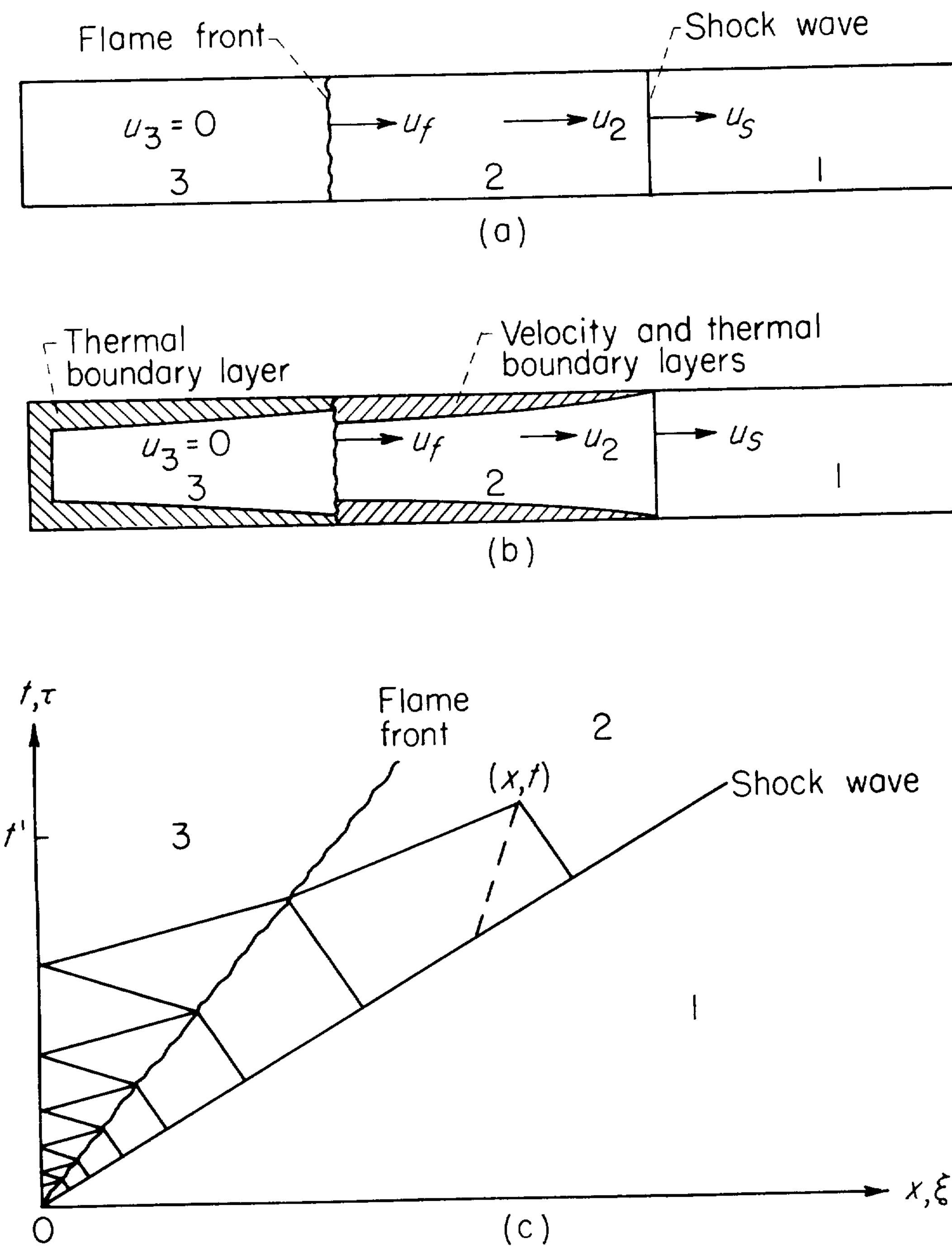
(a) Basic flow at time $t = t'$.
(b) Characteristic diagram.
(c) Characteristic diagram when reflection coefficient at shock is zero.

Figure 5. - Piston-driven shock problem.



(a) Shock tube before diaphragm rupture.
(b) Flow at time $t = t'$.
(c) Characteristics making major contribution to perturbations at (x,t).

Figure 6. - Calculation of nonuniformities and attenuation in shock tubes. Expansion wave assumed to have negligible thickness.



(a) Basic flow (t = t').
(b) Wall boundary layers.
(c) Characteristics for determining perturbations at (x,t). Reflection coefficient at shock assumed negligible.

Figure 7. - Determination of perturbations due to wall boundary layer when planar deflagration is initiated at closed end.